Experience with the Weighted Bootstrap in Testing for Unobserved Heterogeneity in Exponential and Weibull Duration Models*

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Abstract We study the properties of the likelihood-ratio test for unobserved heterogeneity in duration models using mixtures of exponential and Weibull distributions proposed by Cho and White (2010). As they note, this involves a nuisance parameter identified only under the alternative. We apply the asymptotic critical values in Cho and White (2010) and compare these with Hansen's (1996) weighted bootstrap. Our Monte Carlo experiments show that the weighted bootstrap provides superior asymptotic critical values.

Keywords: likelihood-ratio statistic, asymptotic critical values, weighted bootstrap, Monte Carlo experiments.
JEL classification: C12, C41, C80

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1. Introduction

Davies (1977, 1987) first addressed the subject of testing procedures involving parameters not identified under the null. These are now commonly encountered in the modern econometrics literature. For example, Engle and Watson (1987) pointed out that Rosenberg's (1973) conditional heteroskedasticity test involves parameters not identified under the null of conditional homoskedasticity. The generalized autoregressive conditional heteroskedasticity (GARCH) model of Bollerslev (1986) has this feature as well, as specifically examined by Andrews (2001). More recently, Cho and White (2007, 2010) have provided likelihood-ratio (LR) methods for testing for regime switching and unobserved heterogeneity using models with parameters not identified under the null.

The finite sample properties of these tests, and, in particular, the LR test are crucially dependent upon the application of appropriate critical values. In practice, there may be a variety of ways to obtain these, and the various approaches can yield different results. Specifically, in some applications, asymptotic critical values are known and perform quite adequately. In other cases, asymptotic critical values may be unknown, or they may perform poorly. Nevertheless, some form of bootstrap procedure can often give useful critical values in these cases. An especially convenient method applicable to procedures involving nuisance parameters not identified under the null is Hansen's (1996) weighted bootstrap.

Here, our goal is to examine the performance of the LR test for a specific model with nuisance parameters not identified under the null, comparing the use of asymptotic critical values to those obtained using Hansen's (1996) weighted bootstrap. In particular, we undertake extensive, large-scale simulations to investigate the performance of a test for unobserved heterogeneity in duration models proposed by Cho and White (2010). Asymptotic critical values are typically not easy to obtain for such tests, but Cho and White (2010) derive readily computed asymptotic critical values. This creates an unusual opportunity to compare their performance to those obtained from the weighted bootstrap. Our results strongly support the preferred use of the weighted bootstrap in this case. Although there can be no guarantee that these results necessarily generalize to other cases, the strength of the results here and the relative ease of computing the weighted bootstrap support a recommendation to use Hansen's (1996) weighted bootstrap as a default procedure for inference in models with nuisance parameters not identified under the null.

We investigate precisely the same data generating processes (DGPs) and models for the uncensored duration models examined by Cho and White (2010), who test for unobserved heterogeneity using a LR statistic designed to detect discrete mixtures of exponential or Weibull distributions. In Section 2, we briefly discuss the model and the two different methods for constructing critical values. We report the results of Monte Carlo experiments in Section 3, comparing the performances of the asymptotic and weighted bootstrap critical values. Section 4 contains a summary and concluding remarks.

2. Testing for Unobserved Heterogeneity

A conditional Weibull probability model for duration data ($Y_t \in \mathbb{R}^+$) given explanatory variables ($\mathbf{X}_t \in \mathbb{R}^k$) has typical model element

$$f(y \mid \mathbf{X}_t; \delta, \beta, \gamma) \equiv \delta\gamma g(\mathbf{X}_t; \beta) y^{\gamma-1} \exp(-\delta g(\mathbf{X}_t; \beta) y^{\gamma}), \tag{1}$$

for $(\delta, \beta', \gamma) \in D \times B \times \Gamma \subset \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^+$, where $g(\mathbf{X}_t; \cdot)$ is four times continuously differentiable, and δ is identified separately from $g(\mathbf{X}_t; \cdot)$. For example, if $g(\mathbf{X}_t; \beta) = \exp(\mathbf{X}_t'\beta)$ as in Cox (1972), then δ is separately identified. This Weibull model nests the exponential as a special case when $\gamma = 1$.

Cho and White (2010) specify a DGP with possible unobserved heterogeneity having a discrete mixture structure:

$$f_a(y \mid \mathbf{X}_t; \pi^*, \delta_1^*, \delta_2^*, \beta^*, \gamma^*) \equiv \pi^* f(y \mid \mathbf{X}_t; \delta_1^*, \beta^*, \gamma^*) + (1-\pi^*) f(y \mid \mathbf{X}_t; \delta_2^*, \beta^*, \gamma^*), \quad (2)$$

for some unknown $(\pi^*, \delta_1^*, \delta_2^*, \beta^{*\prime}, \gamma^*) \in [0,1] \times D \times D \times B \times \Gamma \subset [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^+$. Heterogeneity is absent if for some $\delta^* \in D$, $\pi^* = 1$ and $\delta_1^* = \delta^*$; $\pi^* = 0$ and $\delta_2^* = \delta^*$; or $\delta_1^* = \delta_2^* = \delta^*$. That is, only δ can be heterogeneous. Cho and White (2010) reparameterize δ_1^* and δ_2^* as $\alpha_1^*\delta^*$ and $\alpha_2^*\delta^*$ respectively, where $\alpha_1^*, \alpha_2^* \in A \equiv \{\alpha : \alpha\delta^* \in D\}$. The null of no heterogeneity and the heterogeneous alternative are then

$$\mathcal{H}_o: \ \pi^* = 1 \text{ and } \alpha_1^* = 1; \alpha_1^* = \alpha_2^* = 1; \text{ or } \pi^* = 0 \text{ and } \alpha_2^* = 1; \text{ versus}$$

$$\mathcal{H}_a: \pi^* \in (0,1) \text{ and } \alpha_1^* \neq \alpha_2^*.$$

Note that any of the conditions in $\mathcal{H}_o$ yields

$$f_a(y \mid \mathbf{X}_t; \pi^*, \delta_1^*, \delta_2^*, \beta^*, \gamma^*) = f(y \mid \mathbf{X}_t; \delta^*, \beta^*, \gamma^*),$$

implying that heterogeneity is absent, and that the null hypothesis arises in three different ways.

The LR statistic for testing unobserved heterogeneity is therefore constructed as

$$LR_n \equiv 2 \left\{ \sum_{t=1}^{n} \ln[f_a(Y_t \mid \mathbf{X}_t; \hat{\pi}_n, \hat{\alpha}_{1n}, \hat{\alpha}_{2n}, \hat{\beta}_{an}, \hat{\gamma}_{an})] - \sum_{t=1}^{n} \ln[f(Y_t \mid \mathbf{X}_t; \hat{\delta}_n, \hat{\beta}_n, \hat{\gamma}_n)] \right\},$$

where n is the sample size, and $(\hat{\delta}_n, \hat{\beta}_n, \hat{\gamma}_n)$ and $(\hat{\pi}_n, \hat{\alpha}_{1n}, \hat{\alpha}_{2n}, \hat{\beta}_{an}, \hat{\gamma}_{an})$ are the maximum-likelihood estimators (MLEs) under the null and alternative, respectively. That is, the

MLEs $(\hat{\delta}_n, \hat{\beta}_n, \hat{\gamma}_n)$ and $(\hat{\pi}_n, \hat{\delta}_{1n}, \hat{\delta}_{2n}, \hat{\beta}_{an}, \hat{\gamma}_{an})$ solve

$$\max_{(\delta,\beta,\gamma)\in D\times B\times\Gamma} \sum_{t=1}^{n} \ln[f(Y_t \mid \mathbf{X}_t; \delta, \beta, \gamma)], \quad \text{and}$$

$$\max_{(\pi,\delta_1,\delta_2,\beta,\gamma)\in[0,1]\times D\times D\times B\times\Gamma} \sum_{t=1}^{n} \ln[f_a(Y_t \mid \mathbf{X}_t; \pi, \delta_1/\hat{\delta}_n, \delta_2/\hat{\delta}_n, \beta, \gamma)],$$

respectively, with $\hat{\alpha}_{1n} \equiv \hat{\delta}_{1n}/\hat{\delta}_n$, $\hat{\alpha}_{2n} \equiv \hat{\delta}_{2n}/\hat{\delta}_n$.

The asymptotic distribution of the LR statistic is non-standard, as there is an unidentified parameter under the null, as well as a boundary parameter. That is, if $\pi^* = 1$ and $\alpha_1^* = 1$ then α_2^* is not identified; if $\alpha_1^* = \alpha_2^* = 1$ then π^* is not identified; and if $\pi^* = 0$ and $\alpha_2^* = 1$ then α_1^* is not identified. As Cho and White (2010) show, LR_n converges in distribution to a function of a Gaussian process under the null. Specifically,

$$LR_n \Rightarrow \mathcal{LR} \equiv \sup_{\alpha\in A} \left(\max[0, \mathcal{G}(\alpha)]\right)^2, \tag{3}$$

where $\mathcal{G}$ is a standard Gaussian process with mean zero and variance one for every α and a covariance structure that differs from case to case. Here, α denotes a representative element of A, and the RHS of (3) is the asymptotic null distribution of the LR test under $\pi^* = 1$ or $\pi^* = 0$, in which α_2 or α_1 is not identified, respectively. By the symmetry of mixture models, the LR test obtained under $\pi^* = 1$ is numerically identical to that obtained under $\pi^* = 0$. We thus avoid any confusion by representing the asymptotic null distribution of the LR test by the generic element α of A.

Cho and White's (2010) theorem 1 derives various covariance structures under exponential and Weibull distribution assumptions, and their theorem 2 shows that these asymptotic null distributions can be obtained by simulating specific Gaussian processes. As is apparent from (3), the asymptotic null distribution of the LR test statistic also depends on A, so we denote the LR test as $LR_n(A)$ to stress the influence of A on the associated inferences. Note that the Gaussian process $\mathcal{G}$ is a function only of α and not δ; this explains why reparameterizing δ_1^* and δ_2^* to $\alpha_1^*\delta^*$ and $\alpha_2^*\delta^*$, respectively, is useful.

An alternative to using the asymptotic critical values is to apply Hansen's (1996) weighted bootstrap. For this, we specify a grid $A_m \subset A$, and for each α in the grid we first compute $\hat{S}_{nt}(\alpha) := \{\hat{D}_n(\alpha)\}^{-\frac{1}{2}}\hat{W}_{nt}(\alpha)$, where

$$\hat{D}_n(\alpha) \equiv \frac{1}{n}\sum_{t=1}^{n}[1-\hat{R}_{nt}(\alpha)]^2 - \frac{1}{n}\sum_{t=1}^{n}[1-\hat{R}_{nt}(\alpha)]\hat{U}_{nt}'\left[\frac{1}{n}\sum_{t=1}^{n}\hat{U}_{nt}\hat{U}_{nt}'\right]^{-1}\frac{1}{n}\sum_{t=1}^{n}\hat{U}_{nt}[1-\hat{R}_{nt}(\alpha)],$$

$$\hat{W}_{nt}(\alpha) \equiv [1-\hat{R}_{nt}(\alpha)] - \hat{U}_{nt}'\left[n^{-1}\sum_{t=1}^{n}\hat{U}_{nt}\hat{U}_{nt}'\right]^{-1}\left[n^{-1}\sum_{t=1}^{n}\hat{U}_{nt}[1-\hat{R}_{nt}(\alpha)]\right],$$

$$\hat{R}_{nt}(\alpha) \equiv f(Y_t|\mathbf{X}_t;\alpha\hat{\delta}_n,\hat{\beta}_n,\hat{\gamma}_n)/f(Y_t|\mathbf{X}_t;\hat{\delta}_n,\hat{\beta}_n,\hat{\gamma}_n), \quad \text{and}$$

$$\hat{U}_{nt} \equiv \nabla_{(\delta,\beta,\gamma)}\ln[f(Y_t|\mathbf{X}_t;\hat{\delta}_n,\hat{\beta}_n,\hat{\gamma}_n)].$$

Observe that $\hat{S}_{nt}(\alpha) := \{\hat{D}_n(\alpha)\}^{-\frac{1}{2}}\hat{W}_{nt}(\alpha)$ is exactly the score function used in the Lagrange multiplier (LM) statistic testing $\pi^* = 1$. That is, $\hat{W}_n(\alpha)$ is the projection error of $\nabla_\pi \ln[f_a(Y_t \mid \mathbf{X}_t;\pi,\alpha_1,\alpha_2,\beta,\gamma)]$ against $\hat{U}_{nt}$, evaluated at the null parameter estimator. Here, α_1 canceles out, but α_2 survives. We replace this with α to avoid confusion, as we explained above. The only difference from the standard score function is that this is now indexed by α to accommodate the fact that α is not identified under the hypothesis $\pi^* = 1$. We can also have exactly the same representation for $\pi^* = 0$ by the symmetry of the mixture, although this is not necessary for obtaining the asymptotic null distribution. $\hat{D}_n(\alpha)$ is a consistent estimator for the asymptotic variance of $\hat{W}_n(\alpha)$.

Second, we generate $\mathcal{Z}_{jt} \sim$ IID $N(0,1)$ ($t = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, J$) to simulate the distribution of the LR statistic as the empirical distribution of

$$\mathcal{LR}_{jn}(A) \equiv \sup_{\alpha\in A_m}\left(\max\left[0, \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\hat{S}_{nt}(\alpha)\mathcal{Z}_{jt}\right]\right)^2.$$

Third, we compare LR_n to this empirical distribution by computing the proportion of simulated outcomes exceeding LR_n. That is, we compute the empirical level $\hat{p}_n \equiv J^{-1} \sum_{j=1}^{J} I[LR_n(A) < \mathcal{LR}_{jn}(A)]$, where $I[\,\cdot\,]$ is the indicator function.

This procedure is essentially the same as that used by Cho and White (2010) to test for unobserved heterogeneity with censored duration data, where asymptotic critical values are not readily available. Cho and White (2010) did not apply the weighted bootstrap for the uncensored duration case, as asymptotic critical values are available there.

The Monte Carlo experiments replicate the above procedure N times, generating $\hat{p}_n^{(i)}, i = 1, ..., N$; and we compute the proportion of outcomes whose $\hat{p}_n^{(i)}$ is less than the specified level (e.g., $\boldsymbol{\alpha} = 5\%$). That is, we compute $N^{-1} \sum_{i=1}^{N} I[\hat{p}_n^{(i)} < \boldsymbol{\alpha}]$. Under the null, this converges to the significance level corresponding to the specified nominal level $\boldsymbol{\alpha}$, whereas this should converge to unity under the alternative.

The intuition for the success of the weighted bootstrap is straightforward. Note that $\hat{S}_{nt}(\cdot)\mathcal{Z}_{jt}$ has a zero population mean function on A due to the fact that $\hat{S}_{nt}(\cdot)$ is independent of $\mathcal{Z}_{jt}$ and that $\mathcal{Z}_{jt}$ is a zero-mean Gaussian random variable. This is true under both $\mathcal{H}_o$ and $\mathcal{H}_a$. Also, $LR_n(A)$ is not bounded in probability under the alternative, so that the chance for $\mathcal{LR}_{jn}(A)$ to be greater than $LR_n(A)$ grows smaller as n increases; this chance is estimated by $\hat{p}_n$. We thus reject the null hypothesis if the empirical level $\hat{p}_n$ is smaller than the specified level of significance $\boldsymbol{\alpha}$. On the other hand, the asymptotic covariance structure of $\hat{S}_{nt}(\cdot)\mathcal{Z}_{jt}$ is identical to that of $\mathcal{G}(\cdot)$ under the null, because $\hat{S}_{nt}(\cdot)$ with the same covariance structure as $\mathcal{G}(\cdot)$ is multiplied by an independent normal variable $\mathcal{Z}_{jt}$ having variance equal to one. Thus, if we draw $\mathcal{LR}_{jn}(A)$ many times by following the steps given above, its distribution asymptotically converges to that of $LR_n(A)$ under the null. That is, $\hat{p}_n$ can estimate the level of significance consistently. For additional discussions, see Cho and White (2010, pp. 473-474) and Hansen (1996).

3. Monte Carlo Experiments

For uncensored duration data, we consider the same DGPs as in Cho and White (2010). For level comparisons, these are:

- $Y_t \sim$ IID Exp(1);
- $Y_t \sim$ IID Weibull$(1, 1)$;
- $Y_t \mid \mathbf{X}_t \sim$ IID Exp$(\exp(\mathbf{X}_t))$;
- $Y_t \mid \mathbf{X}_t \sim$ IID Weibull$(\exp(\mathbf{X}_t), 1)$,

where Exp$(\,\cdot\,)$ and Weibull$(\,\cdot, \cdot\,)$ denote the exponential and Weibull distributions respectively. For the third and fourth DGPs, we let $\mathbf{X}_t \sim$ IID $N(0, 1)$.

These DGPs are estimated using the following parametric models:

- $Y_t \sim$ IID Exp(δ);
- $Y_t \sim$ IID Weibull(δ, γ);
- $Y_t \mid \mathbf{X}_t \sim$ IID Exp$(\delta \exp(\mathbf{X}_t \beta))$;
- $Y_t \mid \mathbf{X}_t \sim$ IID Weibull$(\delta \exp(\mathbf{X}_t \beta), \gamma)$.

We consider nine choices for the domain of α, $A := \{\alpha : \alpha\delta^* \in D\}$: $A =$ [7/9, 2.0], [7/9, 3.0], [7/9, 4.0], [2/3, 2.0], [2/3, 3.0], [2/3, 4.0], [5/9, 2.0], [5/9, 3.0], and [5/9, 4.0]. As mentioned above, the asymptotic distribution of the LR test depends on the properties of A. Our experiments let us examine the impact of the different parameter spaces on the performance of the LR test. Note that the lower bounds of A are now greater than 1/2. Cho and White (2010, p. 461) show that if $\alpha \leq 1/2$, the associated Gaussian process is not defined. This requires defining the lower bound of A be greater than 1/2. Also, we select three upper bounds for A: 2, 3, and 4. These are selected to see how the empirical nominal levels behave as the parameter space A gets larger. These parameter spaces also are the same as in Cho and White (2010).

For power comparisons, we consider the following DGPs:

- $Y_t \mid (\delta_t, \mathbf{X}_t) \sim \text{IID Exp}(\delta_t \exp(\mathbf{X}_t))$;
- $Y_t \mid (\delta_t, \mathbf{X}_t) \sim \text{IID Weibull}(\delta_t \exp(\mathbf{X}_t), 1)$,

where $\mathbf{X}_t \sim \text{IID } N(0, 1)$ as before, and δ_t is a random variable generated by the following various distributions:

- Discrete mixture: $\delta_t \sim \text{IID DM}(0.7370, 1.9296; 0.5)$;
- Gamma mixture: $\delta_t \sim \text{IID Gamma}(5, 5)$;
- Log-normal mixture: $\delta_t \sim \text{IID Log-normal}(-\ln(1.2)/2, \ln(1.2))$;
- Uniform mixture I: $\delta_t \sim \text{IID Uniform}[0.30053, 2.3661]$;
- Uniform mixture II: $\delta_t \sim \text{IID Uniform}[1, 5/3]$,

where $\text{DM}(a, b; p)$ denotes a discrete mixture such that $P[\delta_t = a] = p$ and $P[\delta_t = b] = 1 - p$.

Theorem 2 in Cho and White (2010) justifies obtaining asymptotic critical values by simulating Gaussian processes, and their Monte Carlo experiments show that the asymptotic critical values give correct levels asymptotically and yield tests consistent against the alternative DGPs considered. The empirical rejection rates for critical values corresponding to several nominal levels are provided in Tables 1, 2, 3, and 4 under the null and alternative hypotheses. The results for the 5% nominal level in Tables 1 and 2 are exactly the same as in Table 2 of Cho and White (2010); as our experiments are identical to their experiments, we have not repeated those. Instead, we borrow their results. Here, however, we extend the comparisons to other levels (1% and 10%) to provide a more extensive investigation. For these additional levels, we also find that the empirical rejection rates approach the nominal levels as the sample size n increases. The critical values are conservative, as the approach is from below. We also see that, just as Cho and White (2010) find for the nominal 5% level, as $\inf A$ gets close to $1/2$, the level distortion increases. This is mainly because the desired Gaussian process $\mathcal{G}$ is not defined if $\alpha \leq 1/2$, so that we cannot apply the functional central limit theorem for these parameter values. This also implies that if the associated $\inf A$ is slightly greater than $1/2$, a greater number of observations is required to obtain satisfactory results. This explains the greater level distortions when $\inf A$ is close to $1/2$.

Tables 3 and 4 present empirical rejection rates for the LR test under the alternative using the nominal 5% critical values. These rejection rates are not adjusted for level distortion,[1] so our Tables 3 and 4 differ from tables 3 and 4 of Cho and White (2010). As the conservative nature of the tests should lead us to expect, we see that rejection rates under the alternative are smaller than those for the level-adjusted experiments in Cho and White (2010). As the other findings from these experiments are identical to those in Cho and White (2010), we do not restate them here.

Next, we conduct Monte Carlo experiments using the weighted bootstrap. The simulation results are presented in Tables 5, 6, 7, and 8. The experimental design parameters are identical to those used to analyze censored data in Cho and White (2010). Specifically, we let $J = 500$ and $N = 5,000$ for Tables 5 and 6; and we take $J = 500$ and $N = 2,000$ for Tables 7 and 8.

Tables 5 and 6 correspond to Tables 1 and 2. For all nominal levels, the empirical rejection rates imply conservative inference, as they approach the nominal levels from below as n increases, similar to the previous case. Nevertheless, we see substantive differences from Tables 1 and 2. First, the weighted bootstrap yields empirical rejection rates much closer to the nominal levels than we obtain using the asymptotic critical values. Second, we see much less level distortion as $\inf A$ approaches $1/2$. Third, although

[1] The corresponding rejection rates in Cho and White (2010) are obtained by adjusting levels to remove level distortions. The main focus in Cho and White (2010) is to compare the LR test to the information matrix and Lagrange multiplier tests, which have significant level distortions for small sample sizes. This requires use of level-adjusted critical values for informative comparisons.

the weighted bootstrap works well for mixtures of Weibulls, it works even better for mixtures of exponentials.

Tables 7 and 8 present power performances corresponding to Tables 3 and 4 respectively, again testing at the nominal 5% level. As the sample size increases, the empirical rejection rates approach 100% for every specification, just as in Tables 3 and 4. Nevertheless, we also see differences between the results of Tables 7 and 8 and those of Tables 3 and 4. First, for the mixtures of exponentials, the weighted bootstrap yields better power than using the asymptotic critical values. For small samples ($n = 50$ and 100), the weighted bootstrap always dominates use of the asymptotic critical values. Nevertheless, results for the asymptotic critical values are roughly similar to those for the weighted bootstrap for larger n. Second, for the mixtures of Weibull distributions, the power using the weighted bootstrap is generally better than for the asymptotic critical values, although their behavior is critically dependent upon the parameter space A. When $A = [2/3, 3]$ or $[2/3, 4]$, the asymptotic critical values outperform the weighted bootstrap. On the other hand, when $\inf A$ is close to 0.5 (i.e., when larger A's are considered), the weighted bootstrap performs better. We thus conclude that using the weighted bootstrap is preferable when a relatively larger parameter space A is used. This has practical importance, because researchers are typically unsure about the alternative and thus may tend to choose a larger parameter space to provide greater scope for the alternative.

Finally, Table 9 reports the additional CPU time required to compute weighted bootstrap p-values. These are average CPU times to compute one p-value, obtained by repeating the experiments 10 times. The environments for computing these are identical to the null DGPs reported in Tables 5 and 6. They are computed using GAUSS installed on a 2.39 GHz personal computer. As these are computed using only 10 replications, the results may differ from other simulations conducted in different environments. In particular, when A is large and the sample size is large, we observe large variations in the CPU times. Nevertheless, Table 9 provides enough information to draw some plausible general conclusions. First, the CPU time for the weighted bootstrap increases substantially as the sample size and/or A get larger. Also, increasing the number of explanatory variables $\mathbf{X}_t$ increases the CPU time. Second, Weibull models take more CPU time than exponential models. This is because the Weibull model has more parameters and thus requires more time to compute the associated scores. Third, and significantly, the weighted bootstrap does not demand a substantial amount of CPU time. Given the generally superior performance of the weighted bootstrap as to level, and the resulting improved power, this supports a recommendation that Hansen's (1996) weighted bootstrap be used as a default procedure for testing procedures of the sort considered here.

This conclusion is promising and also suggests a further research topic. Given that the weighted bootstrap can be understood as a generalization of Efron's (1982) bootstrap and that this bootstrap outperforms the asymptotic normal approximation as shown by Bickel and Freedman (1980) and Singh (1981), the simulation results seen here suggest that it may be possible to give conditions under which a similar result holds when bootstrapping random functions instead of random variables. This may require generalizing the regularity conditions of Mason and Newton (1992) who analyze the weighted bootstrap when applied to random numbers. We leave this as a topic for future research.

4. Concluding Remarks

Our goal here is to examine the performance of the LR test for a specific model with nuisance parameters not identified under the null, comparing the use of asymptotic critical values to those obtained using Hansen's (1996) weighted bootstrap. Specifically, we undertake extensive, large-scale simulations to investigate the performance of a test for unobserved heterogeneity in duration models proposed by Cho and White (2010). The availability of Cho and White's (2010) asymptotic critical values for this test makes it possible to compare their performance to critical values obtained from the weighted bootstrap. Our results strongly support the preferred use of the weighted bootstrap in this case. As we noted at the outset, this provides no guarantee that these results necessarily generalize to other cases. Nevertheless, the strength of the results here and the relative ease of computing the weighted bootstrap support a recommendation to use Hansen's (1996) weighted bootstrap as a default procedure for inference in models with nuisance parameters not identified under the null. Our results also motivate future research to examine whether this specific performance holds more generally.

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Table 1. Levels of the LR Test using Asymptotic Critical Values (in Percent)
Number of Repetitions: 10,000

DGP: $Y_t \sim$ IID Exp(1)							
Model: $Y_t \sim \pi\text{Exp}(\delta_1) + (1-\pi)\text{Exp}(\delta_2)$							
Statistics	Level \\n	50	100	500	1,000	2,000	5,000
	1%	0.06	0.10	0.36	0.69	0.72	0.67
$LR_n([7/9,\ 2])$	5%	1.07	1.74	3.35	3.55	4.00	3.88
	10%	3.33	4.86	6.94	7.53	8.61	8.23
	1%	0.19	0.29	0.55	0.76	0.57	0.79
$LR_n([7/9,\ 3])$	5%	1.79	2.59	3.46	3.92	3.76	4.11
	10%	3.96	5.81	7.30	8.31	7.65	8.42
	1%	0.21	0.43	0.54	0.65	0.86	0.82
$LR_n([7/9,\ 4])$	5%	1.77	2.32	3.43	3.62	3.78	4.17
	10%	4.25	5.73	7.39	8.02	8.33	8.52
	1%	0.14	0.25	0.65	0.42	0.72	0.76
$LR_n([2/3,\ 2])$	5%	1.45	2.17	3.28	3.58	3.73	3.75
	10%	3.62	5.06	6.67	7.73	7.74	7.81
	1%	0.31	0.34	0.56	0.60	0.71	0.73
$LR_n([2/3,\ 3])$	5%	1.97	2.26	3.46	3.73	4.11	4.00
	10%	4.72	5.43	7.16	7.26	8.22	8.35
	1%	0.34	0.36	0.69	0.65	0.81	0.80
$LR_n([2/3,\ 4])$	5%	2.14	2.63	3.44	3.48	3.84	4.25
	10%	4.85	6.09	7.25	7.59	8.29	8.71
	1%	0.21	0.25	0.49	0.49	0.78	0.67
$LR_n([5/9,\ 2])$	5%	1.30	1.89	2.79	2.74	3.48	3.73
	10%	3.17	4.29	5.94	6.17	7.27	7.19
	1%	0.24	0.32	0.60	0.50	0.69	0.56
$LR_n([5/9,\ 3])$	5%	1.97	2.38	2.97	2.92	2.98	3.28
	10%	4.11	4.71	6.03	6.44	6.75	6.75
	1%	0.42	0.49	0.55	0.53	0.65	0.66
$LR_n([5/9,\ 4])$	5%	2.12	2.21	2.80	2.98	3.58	3.36
	10%	4.47	5.25	6.19	6.12	6.91	6.87

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DGP: $Y_t \sim$ IID Weibull$(1, 1)$							
Model: $Y_t \sim \pi$Weibull$(\delta_1, \gamma) + (1-\pi)$Weibull$(\delta_2, \gamma)$							
Statistics	Level \ n	50	100	500	1,000	2,000	5,000
	1%	0.00	0.00	0.09	0.18	0.45	0.79
$LR_n([7/9,\ 2])$	5%	0.00	0.13	1.43	2.48	3.54	4.21
	10%	0.25	0.76	4.91	6.55	7.56	8.64
	1%	0.00	0.00	0.04	0.26	0.67	0.82
$LR_n([7/9,\ 3])$	5%	0.01	0.17	1.35	2.91	4.02	4.41
	10%	0.30	0.73	5.03	7.07	8.26	9.01
	1%	0.00	0.01	0.03	0.39	0.73	0.71
$LR_n([7/9,\ 4])$	5%	0.10	0.17	1.57	3.39	3.96	4.02
	10%	0.45	1.02	5.39	7.62	8.25	8.46
	1%	0.00	0.10	0.31	0.57	0.72	0.85
$LR_n([2/3,\ 2])$	5%	0.13	0.29	2.92	3.46	3.74	4.54
	10%	0.90	2.14	6.48	7.10	8.20	8.51
	1%	0.01	0.02	0.49	0.78	0.80	0.82
$LR_n([2/3,\ 3])$	5%	0.26	0.53	3.52	3.92	3.91	4.10
	10%	1.23	2.91	7.77	8.17	7.87	8.54
	1%	0.00	0.01	0.62	0.70	0.79	0.86
$LR_n([2/3,\ 4])$	5%	0.29	0.70	3.43	3.85	4.15	4.13
	10%	1.31	2.79	7.54	7.92	8.12	8.13
	1%	0.01	0.01	0.51	0.57	0.52	0.67
$LR_n([5/9,\ 2])$	5%	0.43	1.17	2.91	3.26	3.18	3.31
	10%	2.24	3.47	6.38	6.62	6.83	7.22
	1%	0.00	0.05	0.73	0.67	0.65	0.63
$LR_n([5/9,\ 3])$	5%	0.50	1.36	3.44	3.41	3.89	3.82
	10%	2.05	4.34	6.56	7.03	7.50	7.54
	1%	0.05	0.08	0.50	0.47	0.85	0.90
$LR_n([5/9,\ 4])$	5%	0.84	1.72	3.48	2.96	4.12	4.04
	10%	2.81	4.91	7.14	6.72	7.70	8.21

Notes: The entries are the empirical rejection rates for the LR statistics under the null hypothesis. For the LR statistics, nine parameter spaces are examined for α: [7/9, 2], [7/9, 3], [7/9, 4], [2/3, 2], [2/3, 3], [2/3, 4], [5/9, 2], [5/9, 3], and [5/9, 4], respectively. The LR statistics are indexed by these spaces, and the entries corresponding to 5% are identical to those in Cho and White (2010).

Table 2. Levels of the Test using Asymptotic Critical Values (in Percent)
Number of Repetitions: 10,000

DGP: $Y_t \mid \mathbf{X}_t \sim$ IID Exp(exp($\mathbf{X}_t$))							
Model: $Y_t \mid \mathbf{X}_t \sim \pi\text{Exp}(\delta_1 \exp(\mathbf{X}_t\beta)) + (1-\pi)\text{Exp}(\delta_2 \exp(\mathbf{X}_t\beta))$							
Statistics	Level \ n	50	100	500	1,000	2,000	5,000
	1%	0.02	0.15	0.43	0.45	0.69	0.64
$LR_n([7/9,\ 2])$	5%	0.68	1.62	2.93	3.32	3.83	4.35
	10%	2.44	4.15	6.61	7.08	8.09	8.73
	1%	0.12	0.21	0.43	0.63	0.66	0.70
$LR_n([7/9,\ 3])$	5%	1.27	2.15	3.10	3.61	3.85	4.17
	10%	3.24	4.70	6.83	7.33	7.93	9.01
	1%	0.16	0.25	0.50	0.65	0.83	0.90
$LR_n([7/9,\ 4])$	5%	1.35	1.71	2.97	3.44	3.89	4.55
	10%	3.55	4.42	7.06	7.71	7.82	8.72
	1%	0.06	0.24	0.43	0.58	0.77	0.75
$LR_n([2/3,\ 2])$	5%	1.14	1.84	2.83	3.24	3.69	3.93
	10%	2.86	4.47	6.16	6.43	7.57	8.03
	1%	0.20	0.41	0.65	0.78	0.71	0.73
$LR_n([2/3,\ 3])$	5%	1.53	2.33	3.33	3.53	3.64	3.74
	10%	3.59	5.17	6.67	7.51	7.50	7.96
	1%	0.24	0.37	0.53	0.75	0.73	0.69
$LR_n([2/3,\ 4])$	5%	1.62	2.26	3.35	3.41	3.71	3.89
	10%	3.81	4.92	6.93	7.45	7.35	8.04
	1%	0.14	0.23	0.47	0.45	0.56	0.59
$LR_n([5/9,\ 2])$	5%	1.13	1.66	2.65	2.89	3.19	3.53
	10%	2.82	3.73	5.58	5.89	6.66	7.38
	1%	0.18	0.36	0.41	0.57	0.63	0.74
$LR_n([5/9,\ 3])$	5%	1.45	1.92	2.31	3.02	3.31	3.32
	10%	3.33	4.22	5.33	6.25	6.62	6.85
	1%	0.17	0.37	0.43	0.45	0.58	0.59
$LR_n([5/9,\ 4])$	5%	1.44	1.99	2.31	3.20	3.31	3.48
	10%	3.36	4.39	5.50	6.66	6.72	7.19

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DGP: $Y_t \mid \mathbf{X}_t \sim$ IID Weibull$(\exp(\mathbf{X}_t), 1)$							
Model: $Y_t \mid \mathbf{X}_t \sim \pi$Weibull$(\delta_1 \exp(\mathbf{X}_t\beta), \gamma) + (1-\pi)$Weibull$(\delta_2 \exp(\mathbf{X}_t\beta), \gamma)$							
Statistics	Level \ n	50	100	500	1,000	2,000	5,000
	1%	0.00	0.00	0.05	0.17	0.54	0.74
$LR_n([7/9,\ 2])$	5%	0.04	0.03	1.25	2.60	3.79	3.80
	10%	0.27	0.64	4.62	6.63	8.21	8.64
	1%	0.00	0.00	0.05	0.17	0.54	0.74
$LR_n([7/9,\ 3])$	5%	0.03	0.12	1.21	3.03	3.74	4.41
	10%	0.32	0.97	4.65	7.34	8.24	8.93
	1%	0.00	0.02	0.03	0.32	0.69	0.92
$LR_n([7/9,\ 4])$	5%	0.07	0.17	1.68	2.96	4.06	4.34
	10%	0.47	0.83	5.18	7.15	8.21	8.64
	1%	0.00	0.00	0.26	0.54	0.83	0.69
$LR_n([2/3,\ 2])$	5%	0.15	0.22	2.72	3.22	4.13	4.05
	10%	0.84	2.16	6.33	7.16	8.13	8.14
	1%	0.00	0.00	0.48	0.69	0.96	0.87
$LR_n([2/3,\ 3])$	5%	0.23	0.46	2.86	3.88	4.20	4.29
	10%	1.29	2.41	6.79	7.76	8.35	8.39
	1%	0.00	0.03	0.36	0.67	0.89	0.68
$LR_n([2/3,\ 4])$	5%	0.33	0.76	3.25	3.70	4.18	4.25
	10%	1.47	3.31	7.31	7.84	7.71	8.44
	1%	0.01	0.05	0.51	0.56	0.57	0.68
$LR_n([5/9,\ 2])$	5%	0.36	1.10	2.76	3.21	3.53	3.39
	10%	1.75	3.94	6.42	6.16	7.20	7.28
	1%	0.03	0.08	0.56	0.64	0.65	0.69
$LR_n([5/9,\ 3])$	5%	0.56	1.51	3.11	3.55	3.38	3.82
	10%	2.58	4.61	6.97	7.29	7.60	7.69
	1%	0.03	0.11	0.60	0.64	0.86	0.67
$LR_n([5/9,\ 4])$	5%	0.86	1.57	3.44	3.50	3.78	3.68
	10%	3.03	4.88	7.31	7.20	7.91	7.61

Notes: The entries are the empirical rejection rates for the LR statistics under the null hypothesis. For the LR statistics, nine parameter spaces are examined for α: [7/9, 2], [7/9, 3], [7/9, 4], [2/3, 2], [2/3, 3], [2/3, 4], [5/9, 2], [5/9, 3], and [5/9, 4], respectively. The LR statistics are indexed by these spaces, and the entries corresponding to 5% are identical to those in Cho and White (2010).

Table 3. Power of the LR Test using Asymptotic Critical Values (Nominal Level: 5%)
Number of Repetitions: 2,000
Model: $Y_t \mid \mathbf{X}_t \sim \text{Exp}(\delta \exp(\mathbf{X}_t \beta))$

Statistics	DGP \ n	50	100	500	1,000	2,000	5,000
	Discrete Mixture	10.95	32.85	96.20	100.0	100.0	100.0
	Gamma Mixture	4.20	14.05	81.80	98.30	100.0	100.0
$LR_n([7/9,\ 2])$	Log-normal Mixture	3.05	10.10	76.60	97.95	100.0	100.0
	Uniform Mixture I	21.15	57.25	99.95	100.0	100.0	100.0
	Uniform Mixture II	1.55	2.50	8.95	12.60	19.20	39.50
	Discrete Mixture	15.40	34.95	96.05	100.0	100.0	100.0
	Gamma Mixture	6.90	15.90	83.55	98.55	100.0	100.0
$LR_n([7/9,\ 3])$	Log-normal Mixture	4.30	12.95	76.65	98.00	100.0	100.0
	Uniform Mixture I	25.35	58.30	99.95	100.0	100.0	100.0
	Uniform Mixture II	2.25	3.40	9.35	12.40	19.25	38.30
	Discrete Mixture	15.80	34.80	95.95	100.0	100.0	100.0
	Gamma Mixture	7.30	16.50	83.10	98.45	100.0	100.0
$LR_n([7/9,\ 4])$	Log-normal Mixture	4.75	13.20	75.45	97.85	100.0	100.0
	Uniform Mixture I	25.50	58.00	99.90	100.0	100.0	100.0
	Uniform Mixture II	2.45	3.50	8.95	12.00	18.55	36.65
	Discrete Mixture	14.25	34.50	96.40	100.0	100.0	100.0
	Gamma Mixture	11.75	31.30	97.50	100.0	100.0	100.0
$LR_n([2/3,\ 2])$	Log-normal Mixture	7.40	22.20	89.65	100.0	100.0	100.0
	Uniform Mixture I	27.30	63.50	99.95	99.50	100.0	100.0
	Uniform Mixture II	1.70	2.35	7.75	11.90	20.95	38.00
	Discrete Mixture	16.60	36.40	96.65	100.0	100.0	100.0
	Gamma Mixture	12.65	31.90	97.40	100.0	100.0	100.0
$LR_n([2/3,\ 3])$	Log-normal Mixture	9.20	22.15	88.85	100.0	100.0	100.0
	Uniform Mixture I	29.40	63.90	99.95	99.45	100.0	100.0
	Uniform Mixture II	2.30	2.27	8.20	12.45	21.00	38.50
	Discrete Mixture	17.00	36.40	96.50	100.0	100.0	100.0
	Gamma Mixture	12.95	31.35	97.40	100.0	100.0	100.0
$LR_n([2/3,\ 4])$	Log-normal Mixture	9.35	21.75	88.65	100.0	100.0	100.0
	Uniform Mixture I	28.90	63.65	99.95	99.45	100.0	100.0
	Uniform Mixture II	2.45	2.80	7.90	12.35	20.85	38.15

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Statistics	DGP \ n	50	100	500	1,000	2,000	5,000
	Discrete Mixture	14.75	32.95	95.70	99.95	100.0	100.0
	Gamma Mixture	15.75	39.55	97.65	99.95	100.0	100.0
$LR_n([5/9,\ 2])$	Log-normal Mixture	10.95	24.75	90.10	99.15	100.0	100.0
	Uniform Mixture I	30.05	61.85	99.65	100.0	100.0	100.0
	Uniform Mixture II	2.10	2.05	6.55	10.25	16.25	36.55
	Discrete Mixture	16.10	34.80	95.75	99.95	100.0	100.0
	Gamma Mixture	16.30	38.35	97.15	99.95	100.0	100.0
$LR_n([5/9,\ 3])$	Log-normal Mixture	11.40	24.55	89.30	99.05	100.0	100.0
	Uniform Mixture I	31.00	61.80	99.75	100.0	100.0	100.0
	Uniform Mixture II	2.60	2.25	6.50	10.50	16.15	36.60
	Discrete Mixture	16.45	34.60	95.70	99.95	100.0	100.0
	Gamma Mixture	16.55	38.05	96.90	99.90	100.0	100.0
$LR_n([5/9,\ 4])$	Log-normal Mixture	11.70	24.65	88.35	98.95	100.0	100.0
	Uniform Mixture I	31.70	61.20	99.70	100.0	100.0	100.0
	Uniform Mixture II	2.60	2.55	6.50	10.05	16.10	34.50

Notes: The entries are the empirical rejection rates for the LR statistics under the five alternative hypothesis: discrete mixture, gamma mixture, log-normal mixture, uniform mixture I, and uniform mixture II. For the LR statistics, nine parameter spaces are examined for α: [7/9, 2], [7/9, 3], [7/9, 4], [2/3, 2], [2/3, 3], [2/3, 4], [5/9, 2], [5/9, 3], and [5/9, 4], respectively. The LR statistics are indexed by these parameter spaces. These entries do not adjust for level distortions, unlike Table 3 of Cho and White (2010).

Table 4. Power of the LR Test using Asymptotic Critical Values (Nominal Level: 5%)
Number of Repetitions: 2,000
Model: $Y_t \mid \mathbf{X}_t \sim \text{Weibull}(\delta \exp(\mathbf{X}_t \beta), \gamma)$

Statistics	DGP \ n	50	100	500	1,000	2,000	5,000
	Discrete Mixture	0.20	1.70	45.35	77.95	97.20	100.0
	Gamma Mixture	0.00	0.05	6.85	42.45	94.85	100.0
$LR_n([7/9,\ 2])$	Log-normal Mixture	0.05	0.05	3.30	23.20	78.85	99.95
	Uniform Mixture I	0.50	5.25	85.50	99.50	100.0	100.0
	Uniform Mixture II	0.30	0.65	2.05	1.80	3.10	6.55
	Discrete Mixture	0.40	2.10	44.05	76.20	96.60	100.0
	Gamma Mixture	0.00	0.00	3.20	31.10	91.70	99.90
$LR_n([7/9,\ 3])$	Log-normal Mixture	0.05	0.00	1.60	15.60	71.05	99.95
	Uniform Mixture I	0.95	4.20	81.30	99.35	100.0	100.0
	Uniform Mixture II	0.20	0.45	1.40	2.10	2.70	5.35
	Discrete Mixture	0.55	2.35	40.90	73.70	96.15	100.0
	Gamma Mixture	0.00	0.00	2.25	24.85	88.95	99.90
$LR_n([7/9,\ 4])$	Log-normal Mixture	0.10	0.00	1.05	11.65	65.55	99.90
	Uniform Mixture I	0.70	3.65	78.15	99.10	100.0	100.0
	Uniform Mixture II	0.10	0.15	1.30	1.90	2.55	5.45
	Discrete Mixture	0.80	4.80	46.10	77.65	96.95	100.0
	Gamma Mixture	0.10	1.10	46.05	89.90	99.70	100.0
$LR_n([2/3,\ 2])$	Log-normal Mixture	0.05	0.75	27.65	66.80	94.00	100.0
	Uniform Mixture I	2.45	13.10	90.75	99.50	100.0	100.0
	Uniform Mixture II	0.40	1.65	5.05	6.70	10.55	18.10
	Discrete Mixture	1.60	5.60	47.05	77.00	96.45	100.0
	Gamma Mixture	0.05	1.05	38.80	87.10	99.60	100.0
$LR_n([2/3,\ 3])$	Log-normal Mixture	0.25	0.55	22.35	62.05	93.05	100.0
	Uniform Mixture I	2.75	12.40	88.25	99.35	100.0	100.0
	Uniform Mixture II	1.05	2.90	5.65	6.95	9.70	17.70
	Discrete Mixture	2.00	5.55	44.20	75.10	95.95	100.0
	Gamma Mixture	0.10	0.70	35.00	85.00	99.55	100.0
$LR_n([2/3,\ 4])$	Log-normal Mixture	0.30	0.65	19.45	57.80	92.15	100.0
	Uniform Mixture I	3.00	10.75	86.85	99.05	100.0	100.0
	Uniform Mixture II	1.20	3.25	5.70	7.40	8.90	16.35

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Statistics	DGP \ n	50	100	500	1,000	2,000	5,000
	Discrete Mixture	1.55	8.40	43.35	74.90	95.85	100.0
	Gamma Mixture	0.75	5.00	64.45	91.90	99.50	99.95
$LR_n([5/9,\ 2])$	Log-normal Mixture	0.40	2.70	40.75	69.85	93.95	100.0
	Uniform Mixture I	4.30	22.60	90.20	99.25	100.0	100.0
	Uniform Mixture II	0.70	2.40	4.35	6.40	9.05	15.10
	Discrete Mixture	3.10	10.10	43.65	74.70	94.95	100.0
	Gamma Mixture	0.65	3.60	56.50	82.75	94.00	98.15
$LR_n([5/9,\ 3])$	Log-normal Mixture	0.45	2.60	35.50	61.40	88.45	99.30
	Uniform Mixture I	2.85	14.40	88.70	99.15	100.0	100.0
	Uniform Mixture II	1.15	2.85	4.60	5.65	7.80	13.40
	Discrete Mixture	3.35	10.30	41.75	73.50	94.60	100.0
	Gamma Mixture	0.75	3.30	55.00	82.10	93.95	98.15
$LR_n([5/9,\ 4])$	Log-normal Mixture	0.55	2.85	34.00	60.15	87.80	99.30
	Uniform Mixture I	4.25	14.65	87.90	98.80	100.0	100.0
	Uniform Mixture II	1.30	2.95	3.85	4.95	7.65	12.05

Notes: The entries are the empirical rejection rates for the LR statistics under the five alternative hypothesis: discrete mixture, gamma mixture, log-normal mixture, uniform mixture I, and uniform mixture II. For the LR statistics, nine parameter spaces are examined for α: [7/9, 2], [7/9, 3], [7/9, 4], [2/3, 2], [2/3, 3], [2/3, 4], [5/9, 2], [5/9, 3], and [5/9, 4], respectively. The LR statistics are indexed by these parameter spaces. These entries do not adjust for level distortions, unlike Table 4 of Cho and White (2010).

Table 5. Bootstrapped Levels of the LR Test (in Percent)
Number of Repetitions: 5,000

DGP: $Y_t \sim$ IID Exp(1)							
Model: $Y_t \sim \pi\text{Exp}(\delta_1) + (1-\pi)\text{Exp}(\delta_2)$							
Statistics	Level $\backslash n$	50	100	500	1,000	2,000	5,000
	1%	0.10	0.16	0.48	0.60	0.74	1.24
$LR_n([7/9,\ 2])$	5%	1.52	2.10	3.52	3.48	4.26	4.84
	10%	4.22	5.86	7.86	8.16	8.40	9.28
	1%	0.26	0.32	0.60	0.70	0.96	0.68
$LR_n([7/9,\ 3])$	5%	2.14	2.60	3.58	3.96	4.58	4.42
	10%	5.24	6.28	7.82	8.34	8.76	9.10
	1%	0.36	0.52	0.90	0.72	0.76	0.80
$LR_n([7/9,\ 4])$	5%	2.88	3.08	4.24	3.96	3.90	4.46
	10%	5.66	6.52	9.00	8.14	8.72	8.50
	1%	0.24	0.28	0.62	0.74	0.64	1.04
$LR_n([2/3,\ 2])$	5%	1.90	2.76	3.68	4.00	3.76	4.30
	10%	5.24	6.14	7.60	8.02	8.12	9.02
	1%	0.34	0.42	0.44	0.76	0.76	0.88
$LR_n([2/3,\ 3])$	5%	2.70	2.88	3.70	3.98	4.48	4.00
	10%	6.44	6.52	8.34	7.90	8.32	8.42
	1%	0.20	0.66	0.56	0.94	1.06	0.84
$LR_n([2/3,\ 4])$	5%	2.50	3.64	3.62	5.10	4.86	3.74
	10%	5.90	7.76	7.92	9.30	9.38	8.38
	1%	0.54	0.44	0.72	0.78	1.00	0.86
$LR_n([5/9,\ 2])$	5%	2.76	2.88	3.38	3.76	4.42	4.52
	10%	5.82	7.02	7.62	8.46	9.02	9.12
	1%	0.46	0.56	0.80	0.66	0.88	0.88
$LR_n([5/9,\ 3])$	5%	2.78	2.98	4.06	4.34	4.18	3.98
	10%	6.58	6.70	8.00	8.90	8.48	8.24
	1%	0.38	0.66	0.90	0.62	0.78	1.02
$LR_n([5/9,\ 4])$	5%	2.78	3.42	4.44	3.50	4.14	4.62
	10%	6.26	7.26	8.60	7.56	8.78	8.94

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DGP: $Y_t \sim$ IID Weibull$(1, 1)$							
Model: $Y_t \sim \pi$Weibull$(\delta_1, \gamma) + (1-\pi)$Weibull$(\delta_2, \gamma)$							
Statistics	Level \ n	50	100	500	1,000	2,000	5,000
$LR_n([7/9,\ 2])$	1%	0.00	0.00	0.04	0.24	0.38	0.86
	5%	0.00	0.10	1.50	2.70	3.54	4.04
	10%	0.48	0.76	5.06	6.98	8.00	8.96
$LR_n([7/9,\ 3])$	1%	0.00	0.00	0.02	0.28	0.62	0.76
	5%	0.02	0.22	1.66	2.72	3.86	3.94
	10%	0.38	1.02	4.98	7.34	7.82	8.64
$LR_n([7/9,\ 4])$	1%	0.00	0.00	0.06	0.30	0.60	0.82
	5%	0.08	0.22	1.54	3.26	4.10	4.16
	10%	0.56	1.04	5.66	7.32	8.94	8.30
$LR_n([2/3,\ 2])$	1%	0.00	0.02	0.36	0.52	0.84	1.02
	5%	0.22	0.88	3.40	3.46	4.16	4.34
	10%	1.74	3.82	8.30	8.32	8.42	8.44
$LR_n([2/3,\ 3])$	1%	0.00	0.04	0.40	0.88	0.76	1.02
	5%	0.26	0.86	3.62	4.16	4.56	4.44
	10%	1.26	3.80	8.12	8.60	8.74	8.98
$LR_n([2/3,\ 4])$	1%	0.00	0.06	0.72	0.88	0.86	0.94
	5%	0.26	0.70	4.12	4.08	3.92	4.44
	10%	1.42	3.40	8.32	8.16	8.20	9.12
$LR_n([5/9,\ 2])$	1%	0.04	0.18	0.50	0.88	0.76	0.88
	5%	1.00	2.22	3.82	4.48	4.50	4.26
	10%	3.98	6.06	8.46	9.12	8.64	8.46
$LR_n([5/9,\ 3])$	1%	0.04	0.12	0.84	0.76	0.68	0.86
	5%	0.88	2.46	4.42	4.58	4.30	4.42
	10%	3.56	6.30	8.26	8.94	8.98	8.52
$LR_n([5/9,\ 4])$	1%	0.02	0.14	0.70	0.82	1.04	1.06
	5%	1.00	2.52	4.16	4.46	4.70	4.56
	10%	3.16	6.36	8.44	9.06	9.16	9.60

Notes: The entries are the empirical rejection rates for the LR statistics under the null hypothesis. For the LR statistics, nine parameter spaces are examined for α: [7/9, 2], [7/9, 3], [7/9, 4], [2/3, 2], [2/3, 3], [2/3, 4], [5/9, 2], [5/9, 3], and [5/9, 4], respectively. The LR statistics are indexed by these spaces.

Table 6. Bootstrapped Levels of the Test (in Percent)
Number of Repetitions: 5,000

DGP: $Y_t \mid \mathbf{X}_t \sim$ IID Exp(exp($\mathbf{X}_t$))							
Model: $Y_t \mid \mathbf{X}_t \sim \pi\text{Exp}(\delta_1 \exp(\mathbf{X}_t\beta)) + (1-\pi)\text{Exp}(\delta_2 \exp(\mathbf{X}_t\beta))$							
Statistics	Level $\backslash n$	50	100	500	1,000	2,000	5,000
	1%	0.12	0.26	0.62	0.52	0.78	0.88
$LR_n([7/9,\ 2])$	5%	0.72	1.88	3.10	2.94	4.12	4.58
	10%	2.62	4.84	6.94	7.12	8.40	8.74
	1%	0.12	0.36	0.66	0.74	0.68	0.92
$LR_n([7/9,\ 3])$	5%	1.64	2.10	3.94	4.00	4.20	4.62
	10%	4.00	4.66	7.46	8.40	8.88	8.28
	1%	0.32	0.36	0.92	0.66	0.72	0.70
$LR_n([7/9,\ 4])$	5%	1.96	2.14	3.92	3.74	3.64	3.72
	10%	4.76	5.28	7.90	8.16	7.84	7.80
	1%	0.18	0.40	0.52	0.72	0.42	0.64
$LR_n([2/3,\ 2])$	5%	1.72	2.80	3.32	3.88	4.08	3.70
	10%	4.00	5.66	7.58	7.74	8.54	8.58
	1%	0.32	0.38	0.56	0.76	0.70	0.86
$LR_n([2/3,\ 3])$	5%	1.86	2.66	4.08	4.14	3.76	4.84
	10%	4.26	6.30	8.14	8.40	7.52	8.82
	1%	0.22	0.28	0.62	0.60	0.88	1.02
$LR_n([2/3,\ 4])$	5%	2.06	2.56	3.64	3.86	3.84	4.56
	10%	4.52	5.74	7.58	8.54	8.12	8.62
	1%	0.14	0.40	0.62	0.52	0.86	1.14
$LR_n([5/9,\ 2])$	5%	1.90	2.52	4.06	3.60	4.18	4.60
	10%	4.74	5.52	7.92	7.88	8.86	9.02
	1%	0.38	0.32	0.78	0.82	1.04	0.82
$LR_n([5/9,\ 3])$	5%	2.04	2.66	3.78	3.80	4.34	4.08
	10%	5.12	5.52	7.74	8.32	8.64	8.22
	1%	0.34	0.52	0.74	0.72	0.98	0.96
$LR_n([5/9,\ 4])$	5%	2.42	3.12	3.96	3.42	4.70	4.12
	10%	5.28	6.42	8.02	7.18	8.74	8.02

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DGP: $Y_t \mid \mathbf{X}_t \sim$ IID Weibull$(\exp(\mathbf{X}_t), 1)$							
Model: $Y_t \mid \mathbf{X}_t \sim \pi$Weibull$(\delta_1 \exp(\mathbf{X}_t\beta), \gamma) + (1-\pi)$Weibull$(\delta_2 \exp(\mathbf{X}_t\beta), \gamma)$							
Statistics	Level \ n	50	100	500	1,000	2,000	5,000
	1%	0.00	0.02	0.06	0.12	0.56	0.82
$LR_n([7/9,\ 2])$	5%	0.04	0.06	1.72	2.66	3.46	4.12
	10%	0.48	0.86	5.62	7.12	7.58	8.70
	1%	0.00	0.00	0.04	0.26	0.56	0.76
$LR_n([7/9,\ 3])$	5%	0.00	0.10	1.40	2.96	3.56	4.42
	10%	0.52	0.82	5.56	7.66	7.88	8.70
	1%	0.00	0.00	0.12	0.48	0.78	0.86
$LR_n([7/9,\ 4])$	5%	0.06	0.16	1.74	3.82	4.38	4.22
	10%	0.44	0.94	4.96	7.58	8.64	8.68
	1%	0.00	0.00	0.36	0.46	0.68	0.64
$LR_n([2/3,\ 2])$	5%	0.26	0.70	2.80	3.48	3.94	4.64
	10%	1.62	3.42	7.12	7.52	8.20	8.76
	1%	0.02	0.00	0.44	0.84	1.00	0.98
$LR_n([2/3,\ 3])$	5%	0.22	0.90	3.60	4.12	4.58	4.08
	10%	1.60	3.26	8.46	8.54	9.14	8.56
	1%	0.00	0.00	0.66	0.76	0.92	0.82
$LR_n([2/3,\ 4])$	5%	0.32	1.06	4.14	4.36	4.46	4.16
	10%	1.64	3.48	8.62	8.56	9.40	8.62
	1%	0.00	0.06	0.64	0.64	0.78	0.78
$LR_n([5/9,\ 2])$	5%	0.64	2.12	3.54	4.22	3.84	4.10
	10%	3.66	6.38	8.24	9.26	8.28	8.46
	1%	0.02	0.08	0.78	0.66	0.90	0.78
$LR_n([5/9,\ 3])$	5%	1.10	2.22	4.22	4.48	4.78	4.46
	10%	3.80	7.14	8.70	8.84	9.10	9.04
	1%	0.10	0.18	0.64	0.96	0.94	0.88
$LR_n([5/9,\ 4])$	5%	0.86	2.50	3.90	4.72	4.22	4.34
	10%	3.32	6.42	8.36	8.90	8.88	9.16

Notes: The entries are the empirical rejection rates for the LR statistics under the null hypothesis. For the LR statistics, nine parameter spaces are examined for α: [7/9, 2], [7/9, 3], [7/9, 4], [2/3, 2], [2/3, 3], [2/3, 4], [5/9, 2], [5/9, 3], and [5/9, 4], respectively. The LR statistics are indexed by these spaces.

Table 7. Bootstrapped Power of the LR Test (Nominal Level: 5%)
Number of Repetitions: 2,000
Model: $Y_t \mid \mathbf{X}_t \sim \text{Exp}(\delta \exp(\mathbf{X}_t \beta))$

Statistics	DGP \ n	50	100	500	1,000	2,000	5,000
	Discrete Mixture	14.90	36.50	97.55	99.95	100.0	100.0
	Gamma Mixture	5.10	15.65	81.20	98.60	100.0	100.0
$LR_n([7/9,\ 2])$	Log-normal Mixture	4.20	10.70	76.20	97.75	100.0	100.0
	Uniform Mixture I	24.25	60.20	99.90	100.0	100.0	100.0
	Uniform Mixture II	2.05	3.65	9.35	13.75	21.60	39.50
	Discrete Mixture	16.60	38.90	96.45	99.85	100.0	100.0
	Gamma Mixture	8.50	18.35	82.45	98.65	100.0	100.0
$LR_n([7/9,\ 3])$	Log-normal Mixture	5.95	14.20	76.60	98.10	100.0	100.0
	Uniform Mixture I	28.00	60.65	99.90	100.0	100.0	100.0
	Uniform Mixture II	2.40	4.25	9.30	13.25	20.20	38.50
	Discrete Mixture	18.95	38.00	96.85	99.90	100.0	100.0
	Gamma Mixture	7.40	17.70	84.15	98.90	100.0	100.0
$LR_n([7/9,\ 4])$	Log-normal Mixture	7.70	13.50	76.90	98.30	100.0	100.0
	Uniform Mixture I	29.55	61.15	99.90	100.0	100.0	100.0
	Uniform Mixture II	2.60	3.75	9.10	14.05	20.75	36.65
	Discrete Mixture	18.80	38.65	96.50	99.85	100.0	100.0
	Gamma Mixture	15.00	35.80	98.20	100.0	100.0	100.0
$LR_n([2/3,\ 2])$	Log-normal Mixture	9.60	25.15	90.35	99.65	100.0	100.0
	Uniform Mixture I	34.35	64.00	99.95	100.0	100.0	100.0
	Uniform Mixture II	2.85	3.60	8.30	13.25	19.75	39.70
	Discrete Mixture	20.55	40.85	97.30	99.95	100.0	100.0
	Gamma Mixture	15.35	33.70	97.80	100.0	100.0	100.0
$LR_n([2/3,\ 3])$	Log-normal Mixture	11.25	26.25	89.00	99.25	100.0	100.0
	Uniform Mixture I	34.15	65.75	99.95	100.0	100.0	100.0
	Uniform Mixture II	3.20	3.90	8.40	13.10	20.70	38.95
	Discrete Mixture	20.65	41.40	97.00	100.0	100.0	100.0
	Gamma Mixture	15.05	34.05	97.70	100.0	100.0	100.0
$LR_n([2/3,\ 4])$	Log-normal Mixture	12.15	22.95	88.55	99.40	100.0	100.0
	Uniform Mixture I	34.10	66.15	99.90	100.0	100.0	100.0
	Uniform Mixture II	3.10	4.40	9.35	14.25	19.85	39.30

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Statistics	DGP \ n	50	100	500	1,000	2,000	5,000
	Discrete Mixture	20.30	38.60	97.30	99.95	100.0	100.0
	Gamma Mixture	22.05	45.20	98.15	100.0	100.0	100.0
$LR_n([5/9,\ 2])$	Log-normal Mixture	15.00	31.50	89.70	99.60	100.0	100.0
	Uniform Mixture I	35.30	66.65	99.95	100.0	100.0	100.0
	Uniform Mixture II	2.65	3.85	10.00	14.65	20.55	37.95
	Discrete Mixture	21.55	42.20	96.60	99.90	100.0	100.0
	Gamma Mixture	19.45	43.60	98.30	100.0	100.0	100.0
$LR_n([5/9,\ 3])$	Log-normal Mixture	15.60	31.20	90.00	99.35	100.0	100.0
	Uniform Mixture I	37.35	67.95	100.0	100.0	100.0	100.0
	Uniform Mixture II	3.80	3.75	9.30	13.10	19.90	38.15
	Discrete Mixture	20.20	40.75	96.75	99.90	100.0	100.0
	Gamma Mixture	21.35	43.30	97.95	100.0	100.0	100.0
$LR_n([5/9,\ 4])$	Log-normal Mixture	14.95	30.65	89.55	99.55	100.0	100.0
	Uniform Mixture I	37.30	64.55	99.90	100.0	100.0	100.0
	Uniform Mixture II	3.40	4.85	9.80	12.95	21.15	36.25

Notes: The entries are the empirical rejection rates for the LR statistics under the five alternative hypothesis: discrete mixture, gamma mixture, log-normal mixture, uniform mixture I, and uniform mixture II. For the LR statistics, nine parameter spaces are examined for α: [7/9, 2], [7/9, 3], [7/9, 4], [2/3, 2], [2/3, 3], [2/3, 4], [5/9, 2], [5/9, 3], and [5/9, 4], respectively. The LR statistics are indexed by these parameter spaces.

Table 8. Bootstrapped Power of the LR Tests (Nominal Level: 5%)
Number of Repetitions: 2,000
Model: $Y_t \mid \mathbf{X}_t \sim \text{Weibull}(\delta \exp(\mathbf{X}_t\beta), \gamma)$

Statistics	DGP \ n	50	100	500	1,000	2,000	5,000
	Discrete Mixture	0.65	2.30	46.25	78.95	97.10	100.0
	Gamma Mixture	0.05	0.20	5.10	38.80	93.70	100.0
$LR_n([7/9,\ 2])$	Log-normal Mixture	0.00	0.10	3.85	25.60	78.15	99.95
	Uniform Mixture I	0.75	4.95	86.65	99.70	100.0	100.0
	Uniform Mixture II	0.20	1.05	4.90	6.85	9.80	16.75
	Discrete Mixture	0.65	2.50	44.30	76.30	97.15	100.0
	Gamma Mixture	0.00	0.00	2.75	27.65	88.40	100.0
$LR_n([7/9,\ 3])$	Log-normal Mixture	0.05	0.05	1.90	16.10	67.95	99.85
	Uniform Mixture I	0.30	4.45	80.85	98.95	100.0	100.0
	Uniform Mixture II	0.50	1.65	5.30	7.55	9.85	15.75
	Discrete Mixture	0.85	2.55	41.65	73.20	96.50	100.0
	Gamma Mixture	0.05	0.20	2.30	19.05	82.80	99.45
$LR_n([7/9,\ 4])$	Log-normal Mixture	0.05	0.05	1.95	10.15	61.80	99.65
	Uniform Mixture I	0.50	3.35	77.75	99.10	100.0	100.0
	Uniform Mixture II	0.55	1.60	4.70	7.10	9.15	16.20
	Discrete Mixture	1.55	6.55	50.10	78.60	97.50	100.0
	Gamma Mixture	0.30	1.55	48.55	90.45	99.65	100.0
$LR_n([2/3,\ 2])$	Log-normal Mixture	0.45	0.90	29.90	68.70	94.85	99.95
	Uniform Mixture I	3.50	14.60	90.40	99.60	100.0	100.0
	Uniform Mixture II	0.65	2.20	5.40	7.60	10.55	18.95
	Discrete Mixture	1.65	6.85	50.85	77.75	95.75	100.0
	Gamma Mixture	0.25	0.80	37.95	87.40	99.70	100.0
$LR_n([2/3,\ 3])$	Log-normal Mixture	0.10	0.65	21.85	62.25	92.90	99.95
	Uniform Mixture I	2.65	11.90	88.20	99.20	100.0	100.0
	Uniform Mixture II	0.75	2.95	5.80	7.55	9.95	16.90
	Discrete Mixture	1.50	7.10	47.65	74.00	95.00	100.0
	Gamma Mixture	0.30	0.90	29.70	80.25	98.00	100.0
$LR_n([2/3,\ 4])$	Log-normal Mixture	0.20	0.55	20.00	53.40	91.05	100.0
	Uniform Mixture I	2.70	10.30	85.90	99.30	100.0	100.0
	Uniform Mixture II	1.40	2.65	6.60	6.80	8.60	16.15

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Statistics	DGP \ n	50	100	500	1,000	2,000	5,000
	Discrete Mixture	4.20	11.85	49.90	80.10	96.75	100.0
	Gamma Mixture	1.90	7.55	72.20	94.00	99.85	100.0
$LR_n([5/9,\ 2])$	Log-normal Mixture	2.30	6.20	45.20	74.40	95.40	99.90
	Uniform Mixture I	8.95	25.60	89.90	99.30	100.0	100.0
	Uniform Mixture II	1.65	2.75	5.10	8.30	10.60	18.90
	Discrete Mixture	4.85	11.65	48.30	76.55	96.45	100.0
	Gamma Mixture	1.30	4.75	65.05	92.05	99.85	100.0
$LR_n([5/9,\ 3])$	Log-normal Mixture	0.85	3.75	39.80	70.45	94.40	100.0
	Uniform Mixture I	5.95	22.05	87.65	99.30	100.0	100.0
	Uniform Mixture II	1.50	3.70	6.45	7.55	10.90	17.45
	Discrete Mixture	4.50	10.75	46.50	74.60	95.45	99.95
	Gamma Mixture	1.55	3.70	59.10	88.90	98.90	100.0
$LR_n([5/9,\ 4])$	Log-normal Mixture	1.15	2.25	34.95	67.70	92.05	99.95
	Uniform Mixture I	5.45	17.45	86.50	99.25	100.0	100.0
	Uniform Mixture II	2.40	4.00	6.20	6.50	11.15	15.30

Notes: The entries are the empirical rejection rates for the LR statistics under the five alternative hypothesis: discrete mixture, gamma mixture, log-normal mixture, uniform mixture I, and uniform mixture II. For the LR statistics, nine parameter spaces are examined for α: [7/9, 2], [7/9, 3], [7/9, 4], [2/3, 2], [2/3, 3], [2/3, 4], [5/9, 2], [5/9, 3], and [5/9, 4], respectively. The LR statistics are indexed by these parameter spaces.

Table 9. Additional CPU Times for Weighted Bootstrapping (in Seconds)

DGP: $Y_t \sim$ IID Exp(1)						
Model: $Y_t \sim \pi\text{Exp}(\delta_1) + (1-\pi)\text{Exp}(\delta_2)$						
Statistics \ n	50	100	500	1,000	2,000	5,000
$LR_n([7/9,\ 2])$	0.35	0.40	1.15	1.99	6.37	21.48
$LR_n([7/9,\ 3])$	0.64	0.75	2.08	3.38	7.22	36.35
$LR_n([7/9,\ 4])$	0.93	1.05	3.21	5.33	17.53	51.78
$LR_n([2/3,\ 2])$	0.39	0.44	1.21	2.20	6.48	22.97
$LR_n([2/3,\ 3])$	0.67	0.75	2.33	3.92	10.94	37.78
$LR_n([2/3,\ 4])$	0.96	1.09	3.60	6.32	19.22	58.19
$LR_n([5/9,\ 2])$	0.41	0.49	1.44	2.21	7.98	25.62
$LR_n([5/9,\ 3])$	0.70	0.79	2.43	6.13	11.25	39.09
$LR_n([5/9,\ 4])$	0.99	1.13	3.62	6.52	20.28	60.16
DGP: $Y_t \sim$ IID Weibull$(1,1)$						
Model: $Y_t \sim \pi\text{Weibull}(\delta_1,\gamma) + (1-\pi)\text{Weibull}(\delta_2,\gamma)$						
Statistics \ n	50	100	500	1,000	2,000	5,000
$LR_n([7/9,\ 2])$	0.50	0.64	1.82	4.31	8.26	25.51
$LR_n([7/9,\ 3])$	0.91	1.17	3.36	7.75	17.28	48.02
$LR_n([7/9,\ 4])$	1.29	1.64	5.16	12.01	22.97	71.97
$LR_n([2/3,\ 2])$	0.54	0.70	1.98	4.77	11.08	28.04
$LR_n([2/3,\ 3])$	0.93	1.22	3.68	8.67	17.97	55.37
$LR_n([2/3,\ 4])$	1.33	1.71	5.43	12.70	25.22	78.59
$LR_n([5/9,\ 2])$	0.54	0.75	2.08	5.29	10.51	28.80
$LR_n([5/9,\ 3])$	0.97	1.28	3.96	8.69	18.56	58.96
$LR_n([5/9,\ 4])$	1.40	1.81	5.70	12.70	28.99	79.51

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DGP: $Y_t \mid \mathbf{X}_t \sim$ IID Exp(exp($\mathbf{X}_t$))						
Model: $Y_t \mid \mathbf{X}_t \sim \pi\text{Exp}(\delta_1 \exp(\mathbf{X}_t\beta)) + (1-\pi)\text{Exp}(\delta_2 \exp(\mathbf{X}_t\beta))$						
Statistics \ n	50	100	500	1,000	2,000	5,000
$LR_n([7/9,\ 2])$	0.47	0.61	1.83	3.88	8.99	23.21
$LR_n([7/9,\ 3])$	0.87	1.14	3.40	7.14	16.70	46.19
$LR_n([7/9,\ 4])$	1.24	1.60	5.10	9.98	23.42	66.50
$LR_n([2/3,\ 2])$	0.53	0.67	2.00	4.50	10.20	28.02
$LR_n([2/3,\ 3])$	0.90	1.14	3.81	7.24	18.69	50.29
$LR_n([2/3,\ 4])$	1.27	1.62	5.41	10.49	24.57	68.95
$LR_n([5/9,\ 2])$	0.56	0.73	2.05	4.94	11.05	30.48
$LR_n([5/9,\ 3])$	0.93	1.22	4.10	7.95	19.81	55.72
$LR_n([5/9,\ 4])$	1.29	1.66	5.48	11.28	26.98	79.41
DGP: $Y_t \mid \mathbf{X}_t \sim$ IID Weibull(exp($\mathbf{X}_t$), 1)						
Model: $Y_t \mid \mathbf{X}_t \sim \pi\text{Weibull}(\delta_1 \exp(\mathbf{X}_t\beta), \gamma) + (1-\pi)\text{Weibull}(\delta_2 \exp(\mathbf{X}_t\beta), \gamma)$						
Statistics \ n	50	100	500	1,000	2,000	5,000
$LR_n([7/9,\ 2])$	0.52	0.68	2.33	6.00	12.10	22.72
$LR_n([7/9,\ 3])$	0.95	1.27	5.86	10.80	24.10	35.87
$LR_n([7/9,\ 4])$	1.36	1.83	6.59	14.85	33.44	57.61
$LR_n([2/3,\ 2])$	0.60	0.75	3.52	6.23	15.14	28.62
$LR_n([2/3,\ 3])$	1.02	1.28	6.00	10.72	25.30	42.92
$LR_n([2/3,\ 4])$	1.48	1.83	7.25	15.73	36.45	96.10
$LR_n([5/9,\ 2])$	0.67	0.82	3.82	6.88	15.41	31.13
$LR_n([5/9,\ 3])$	1.06	1.36	6.12	11.97	27.58	59.05
$LR_n([5/9,\ 4])$	1.49	1.86	8.72	14.12	36.58	102.5

Notes: The entries are the additional times for conducting the weighted bootstrap for the LR statistics indexed by the nine parameter spaces for α, [7/9, 2], [7/9, 3], [7/9, 4], [2/3, 2], [2/3, 3], [2/3, 4], [5/9, 2], [5/9, 3], and [5/9, 4].