Mathematical Proofs for
"Testing Linearity Using Power Transforms of Regressors"

YAE IN BAEK
Department of Economics
University of California, San Diego
9500 Gilman Dr. La Jolla, CA, 92093, USA

JIN SEO CHO
School of Economics
Yonsei University
50 Yonsei-ro, Seodaemun-gu, Seoul, 120-749, Korea

PETER C.B. PHILLIPS
Yale University, University of Auckland
Singapore Management University & University of Southampton

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Abstract

We provide mathematical proofs for the results in "Testing Linearity Using Power Transforms of Regressors" by Baek, Cho, and Phillips (2014).

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1 Introduction

This note provides mathematical proofs of the results stated Baek, Cho, and Phillips (2014). We avoid possible confusions by using an equation number system different from that in Baek, Cho, and Phillips (2014) using square brackets.

1 Proofs

Proof of Theorem 1: It is elementary to show that $\text{plim}_{n\to\infty}\widehat{\sigma}^2_{n,0} = \sigma^2_*$. We therefore focus on the numerator and denominator of (2) separately. The scaled numerator is $n^{-1/2}\mathbf{X}(\cdot)'\mathbf{M}\mathbf{U}$ and the uniform law of large numbers (ULLN) can be applied to $\{n^{-1}\sum_{t=1}^{n} X_t^{\gamma}\mathbf{Z}_t\}$, so that for each $j = 1, 2, \ldots, 2+k$,

$$\sup_{\gamma\in\mathbf{\Gamma}}\left|n^{-1}\sum_{t=1}^{n} X_t^{\gamma}Z_{t,j} - \mathbb{E}[X_t^{\gamma}Z_{t,j}]\right| \stackrel{\mathbb{P}}{\to} 0, \tag{A.1}$$

where $Z_{t,j}$ is the j-th row element of $\mathbf{Z}_t$. This result mainly follows from theorem 3(a) of Andrews (1992). In particular, Assumption 1(*ii*) implies that $\mathbf{\Gamma}$ is totally bounded; for $j = 1, 2, \cdots, k+2$, $\mathbb{E}[|X_t^{\gamma}Z_{t,j}|] \leq \mathbb{E}[M_t^2] < \infty$ by Assumption 2, so that for each $\gamma \in \mathbf{\Gamma}$, the ergodic theorem holds for $n^{-1}\sum_{t=1}^{n} X_t^{\gamma}\ Z_{t,j}$; and finally $X_t^{(\cdot)}Z_{t,j}$ is Lipschitz continuous because for each j,

$$|X_t^{\gamma}Z_{t,j} - X_t^{\gamma'}Z_{t,j}| \leq \sup_{\gamma\in\mathbf{\Gamma}}|X_t^{\gamma}\log(X_t)|\cdot|Z_{t,j}|\cdot|\gamma-\gamma'| \leq M_t^2|\gamma-\gamma'|, \tag{A.2}$$

where $M_t^2 = O_p(1)$. These three conditions are the assumptions required in theorem 3(a) of Andrews (1992) to prove the ULLN. This also implies that $\mathbb{E}[X_t^{(\cdot)}\mathbf{Z}_t]$ is continuous on $\mathbf{\Gamma}$. Since $n^{-1}\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}_t' \stackrel{\mathbb{P}}{\to} \mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']$ by ergodicity we obtain $\sup_{\gamma\in\mathbf{\Gamma}}\left|n^{-1/2}\mathbf{X}(\gamma)'\mathbf{M}\mathbf{U} - n^{-1/2}\{\mathbf{X}(\gamma)'\mathbf{U} - \mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbf{Z}'\mathbf{U}\}\right| = o_p(1)$. Given this, it follows that $n^{-1/2}\{\mathbf{X}(\cdot)'\mathbf{U} - \mathbb{E}[X_t^{(\cdot)}\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbf{Z}'\mathbf{U}\} \Rightarrow \mathcal{G}(\cdot)$, where $\mathcal{G}(\cdot)$ is a zero mean Gaussian process with the covariance kernel $\kappa(\cdot,\cdot)$. For this, we apply the central limit theorem (CLT) to $n^{-1/2}\mathbf{Z}'\mathbf{U}$, so that $n^{-1/2}\mathbf{Z}'\mathbf{U} \stackrel{\text{A}}{\sim} N(\mathbf{0}, \mathbb{E}[U_t^2\mathbf{Z}_t\mathbf{Z}_t'])$. Next, $X_t^{(\cdot)}U_t$ is Lipschitz continuous, so that $|X_t^{\gamma}U_t - X_t^{\gamma'}U_t| \leq \sup_{\gamma\in\mathbf{\Gamma}}|X_t^{\gamma}\log(X_t)|\cdot|U_t|\cdot|\gamma-\gamma'| \leq M_t^2|\gamma-\gamma'|$ by Assumption 2, implying that $\mathbb{E}[\sup_{|\gamma-\gamma'|\leq\eta}|X_t^{\gamma}U_t - X_t^{\gamma'}U_t|^{2r}]^{\frac{1}{2r}} \leq \mathbb{E}[M_t^{4r}]^{\frac{1}{2r}}\eta$. This implies that $\{n^{-1/2}\mathbf{X}(\cdot)'\mathbf{U}\}$ is tight because Ossiander's L^{2r} entropy is finite by theorem 1 of Doukhan, Massart, and Rio (1995). We further note that (A.2) implies that for some $c > 0$, $\mathbb{E}[\sup_{|\gamma-\gamma'|<\eta}|\mathbb{E}[(X_t^{\gamma} - X_t^{\gamma'})\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbf{Z}_tU_t|^{2r}]^{\frac{1}{2r}} \leq c\mathbb{E}[M_t^{4r}]^{\frac{1}{2r}}\mathbb{E}[M_t^2]\eta$, so that $\{n^{-1/2}\mathbb{E}[X_t^{(\cdot)}\mathbf{Z}_t']\ \mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbf{Z}'\mathbf{U}\}$ is tight. Hence $\{n^{-1/2}(\mathbf{X}(\cdot)'\mathbf{U} - \mathbb{E}[X_t^{(\cdot)}\mathbf{Z}_t']\ \mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbf{Z}'\mathbf{U})\}$ is also tight. Furthermore, the finite-dimensional multivariate CLT holds by the martingale CLT. It follows that $n^{-1/2}\{\mathbf{X}(\cdot)'\mathbf{U} - \mathbb{E}[X_t^{(\cdot)}\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbf{Z}'\mathbf{U}\} \Rightarrow \mathcal{G}(\cdot)$, implying that $n^{-1/2}\mathbf{X}(\cdot)'\mathbf{M}\mathbf{U} \Rightarrow \mathcal{G}(\cdot)$.

Second, we apply the ULLN to $n^{-1}\mathbf{X}(\cdot)'\mathbf{M}\mathbf{X}(\cdot)$. We separate our proof into two parts: we first show that $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{X}(\gamma)'\mathbf{X}(\gamma) - \mathbb{E}[X_t^{2\gamma}]| = o_p(1)$ and next show that $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{X}(\gamma)'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}(\gamma) - \mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbb{E}[\mathbf{Z}_t\mathbf{X}_t^{\gamma}]| = o_p(1)$. It then follows that $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{X}(\gamma)'\mathbf{M}\mathbf{X}(\gamma) - \mathbb{E}[X_t^{2\gamma}] + (\mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbb{E}[\mathbf{Z}_t X_t^{\gamma}])| = o_p(1)$. For this goal, we first note that $X_t^{2(\cdot)}$ is Lipschitz continuous, so that $|X_t^{2\gamma} - X_t^{2\gamma'}| \leq 2\sup_{\gamma\in\mathbf{\Gamma}} |X_t^{2\gamma}\log(X_t)| \cdot |\gamma - \gamma'| \leq 2\sup_{\gamma\in\mathbf{\Gamma}} |X_t^{\gamma}\log(X_t)| \cdot \sup_{\gamma\in\mathbf{\Gamma}} |X_t^{\gamma}| \cdot |\gamma-\gamma'| \leq 2M_t^2|\gamma - \gamma'|$, and $2M_t^2 = O_p(1)$ by Assumption 2. Theorem 3 of Andrews (1992) now shows that the ULLN holds for $\{n^{-1}\sum_{t=1}^{n} X_t^{2(\cdot)} - \mathbb{E}[X_t^{2(\cdot)}]\}$. We next note that

$$
\begin{aligned}
&\sup_{\gamma\in\mathbf{\Gamma}} \left|n^{-1}\mathbf{X}(\gamma)'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}(\gamma) - \mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbb{E}[\mathbf{Z}_t\mathbf{X}_t^{\gamma}]\right| \\
&\quad \leq \sup_{\gamma\in\mathbf{\Gamma}} \left|(n^{-1}\mathbf{X}(\gamma)'\mathbf{Z} - \mathbb{E}[X_t^{\gamma}\mathbf{Z}_t'])(n^{-1}\mathbf{Z}'\mathbf{Z})^{-1}n^{-1}\mathbf{Z}'\mathbf{X}(\gamma)\right| \\
&\qquad + \sup_{\gamma\in\mathbf{\Gamma}} \left|\mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']((n^{-1}\mathbf{Z}'\mathbf{Z})^{-1} - \mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1})n^{-1}\mathbf{Z}'\mathbf{X}(\gamma)\right| \\
&\qquad\quad + \sup_{\gamma\in\mathbf{\Gamma}} \left|\mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}(n^{-1}\mathbf{Z}'\mathbf{X}(\gamma) - \mathbb{E}[\mathbf{Z}_t X_t^{\gamma}])\right|.
\end{aligned}
$$

Hence, $\sup_{\gamma\in\mathbf{\Gamma}} |(n^{-1}\mathbf{X}(\gamma)'\mathbf{Z}) - \mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']| = o_p(1)$ by (A.1), and $(n^{-1}\mathbf{Z}'\mathbf{Z})^{-1} - \mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1} = o_p(1)$ by Assumption 2 and ergodicity. Furthermore, $\sup_{\gamma\in\mathbf{\Gamma}} \left|n^{-1}\mathbf{X}(\gamma)'\mathbf{Z}\right| = O_p(1)$ by Assumption 2, so that $\sup_{\gamma\in\mathbf{\Gamma}} |\mathbb{E}[X_t^{\gamma}\mathbf{Z}_t]| = O(1)$. Therefore, $\sup_{\gamma\in\mathbf{\Gamma}} |(n^{-1}\mathbf{X}(\gamma)'\mathbf{Z} - \mathbb{E}[X_t^{\gamma}\mathbf{Z}_t'])(n^{-1}\mathbf{Z}'\mathbf{Z})^{-1}n^{-1}\mathbf{Z}'\mathbf{X}(\gamma)| \leq \sup_{\gamma\in\mathbf{\Gamma}} \left|(n^{-1}\mathbf{X}(\gamma)'\mathbf{Z} - \mathbb{E}[X_t^{\gamma}\mathbf{Z}_t'])\right| \cdot \left|(n^{-1}\mathbf{Z}'\mathbf{Z})^{-1}\right| \cdot \sup_{\gamma\in\mathbf{\Gamma}} \left|n^{-1}\mathbf{Z}'\mathbf{X}(\gamma)\right| = o_p(1)$, where for an arbitrary function $\mathbf{f}(x) := [f_{i,j}(x)]$, we let $\sup_x |\mathbf{f}(x)| := [\sup_x |f_{i,j}(x)|]$. In a similar manner, it also follows that $\sup_{\gamma\in\mathbf{\Gamma}} |\mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']((n^{-1}\mathbf{Z}'\mathbf{Z})^{-1} - \mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1})n^{-1}\mathbf{Z}'\mathbf{X}(\gamma)| \leq \sup_{\gamma\in\mathbf{\Gamma}} |\mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']| \cdot |((n^{-1}\mathbf{Z}'\mathbf{Z})^{-1} - \mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1})| \cdot \sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{Z}'\mathbf{X}(\gamma)| = o_p(1)$ and $\sup_{\gamma\in\mathbf{\Gamma}} \left|\mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}(n^{-1}\mathbf{Z}'\mathbf{X}(\gamma) - \mathbb{E}[\mathbf{Z}_t X_t^{\gamma}])\right| \leq \sup_{\gamma\in\mathbf{\Gamma}} |\mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']| \cdot \left|\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\right| \sup_{\gamma\in\mathbf{\Gamma}} \left|(n^{-1}\mathbf{Z}'\mathbf{X}(\gamma) - \mathbb{E}[\mathbf{Z}_t X_t^{\gamma}])\right| = o_p(1)$. These two facts imply that $\sup_{\gamma\in\mathbf{\Gamma}} \left|n^{-1}\mathbf{X}(\gamma)'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}(\gamma) - \mathbb{E}[X_t^{\gamma}\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbb{E}[\mathbf{Z}_t\mathbf{X}_t^{\gamma}]\right| = o_p(1)$. Use of the continuous mapping theorem completes the proof. ■

Before proving Theorem 2, we provide supplementary lemmas to assist in proving the main claim more efficiently.

Lemma A1. *Given Assumptions 1 and 3,*

(i) $\mathbf{L}_1'\mathbf{U} = O_p(\sqrt{n})$, $\mathbf{Z}'\mathbf{U} = O_p(\sqrt{n})$, $\mathbf{K}_1'\mathbf{U} = O_p(\sqrt{n})$, *where for* $j = 1, 2, \ldots$, $\mathbf{K}_j := [\mathbf{L}_j \vdots \mathbf{0}_{n\times k}]$;

(ii) $\mathbf{L}_1'\mathbf{Z} = O_p(n)$, $\mathbf{Z}'\mathbf{Z} = O_p(n)$, $\mathbf{K}_1'\mathbf{Z} = O_p(n)$;

(iii) $\mathbf{L}_1'\mathbf{L}_1 = O_p(n)$, $\mathbf{L}_1'\mathbf{K}_1 = O_p(n)$, $\mathbf{L}_2'\mathbf{U} = O_p(n)$, $\mathbf{L}_2'\mathbf{Z} = O_p(n)$, $\mathbf{K}_1'\mathbf{Z} = O_p(n)$, $\mathbf{K}_1'\mathbf{K}_1 = O_p(n)$, $\mathbf{K}_2'\mathbf{U} = O_p(n)$, *and* $\mathbf{K}_2'\mathbf{Z} = O_p(n)$; *and*

(iv) $\mathbf{L}_2'\mathbf{U} = o_p(n)$ *and* $\mathbf{K}_2'\mathbf{U} = o_p(n)$. □

Lemma A2. *Given Assumptions 1, 3, and* $\mathcal{H}_0''$,

(i) $L_n^{(1)}(0;\alpha) = 2(\alpha_* - \alpha)\mathbf{L}_1'\mathbf{M}\mathbf{U} + 2\mathbf{U}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} - \mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$, *where for* $j = 1, 2, \ldots$, $L_n^{(j)}(0;\alpha) := (\partial^j/\partial\gamma^j)L_n(\gamma;\alpha)|_{\gamma=0}$;

(ii) $L_n^{(1)}(0;\alpha) = 2(\alpha_* - \alpha)\mathbf{L}_1'\mathbf{M}\mathbf{U} + o_p(\sqrt{n})$; *and*

(iii) $L_n^{(2)}(0;\alpha) = -2(\alpha_* - \alpha)^2\mathbf{L}_1'\mathbf{M}\mathbf{L}_1 + o_p(n)$. □

Lemma A3. *Given Assumptions 1, 3, and* $\mathcal{H}_0''$,

(i) $QLR_n^{(\gamma=0;\beta)} = \{\mathbf{L}_1'\mathbf{M}\mathbf{U}\}^2/\{\hat{\sigma}_{n,0}^2(\mathbf{L}_1'\mathbf{M}\mathbf{L}_1)\} + o_p(1)$; *and*

(ii) $QLR_n^{(\gamma=0;\beta)} = O_p(1)$. □

Lemma A4. *Given Assumptions 1, 3, and* $\mathcal{H}_0''$,

(i) $QLR_n^{(\gamma=0;\alpha)} = \{\mathbf{L}_1'\mathbf{M}\mathbf{U}\}^2/\{\hat{\sigma}_{n,0}^2(\mathbf{L}_1'\mathbf{M}\mathbf{L}_1)\} + o_p(1)$; *and*

(ii) $QLR_n^{(\gamma=0;\alpha)} = O_p(1)$. □

Proof of Lemma A1: (*i*) By the definition of $\mathbf{K}_1$, we note that if $\mathbf{L}_1'\mathbf{U} = O_p(\sqrt{n})$, $\mathbf{K}_1'\mathbf{U} = O_p(\sqrt{n})$. We, therefore, focus on proving that $\mathbf{L}_1'\mathbf{U} = O_p(\sqrt{n})$ and $\mathbf{Z}'\mathbf{U} = O_p(\sqrt{n})$. We also note that the structures of $\mathbf{L}_1'\mathbf{U}$ and $\mathbf{Z}'\mathbf{U}$ are identical. Accordingly, we let $\mathbf{R}$ be generic notation for $\mathbf{L}_1$ and $\mathbf{Z}$ and prove the given claims using $\mathbf{R}'\mathbf{U}$.

If we let $\mathbf{R} = [\mathbf{R}_{tj}]$, $\mathbf{R}'\mathbf{U} = \sum R_{tj}U_t$, which obeys the CLT if $\mathbb{E}[R_{tj}^2U_t^2] < \infty$. We note that $\mathbb{E}[R_{tj}^2U_t^2] \leq \mathbb{E}[R_{tj}^4]^{1/2}\mathbb{E}[U_t^4]^{1/2}$ by Cauchy-Schwarz, so the desired result follows since $\mathbb{E}[Z_{tj}^4] < \infty$, $\mathbb{E}[\log^4(X_t)] < \infty$, and $\mathbb{E}[U_t^4] < \infty$ by Assumption 3.

(*ii*) As in (*i*), if $\mathbf{L}_1'\mathbf{Z} = O_p(n)$, $\mathbf{K}_1'\mathbf{Z} = O_p(n)$ by the definition of $\mathbf{K}_1$. As before, we let $\mathbf{R}$ be generic notation for $\mathbf{L}_1$ and $\mathbf{Z}$ and prove the given claims using $\mathbf{R}'\mathbf{Z}$. As $\mathbf{R}'\mathbf{Z} = [\sum R_{tj}Z_{ti}]$, the result follows by ergodicity if $\mathbb{E}[|R_{tj}Z_{ti}|] < \infty$, which holds by virtue of Cauchy-Schwarz and the fact that $\mathbb{E}[\log^2(X_t)] < \infty$ and $\mathbb{E}[Z_{ti}^2] < \infty$ by Assumption 3.

(*iii*) By the definitions of $\mathbf{K}_1$ and $\mathbf{K}_2$, if $\mathbf{L}_1'\mathbf{L}_1 = O_p(n)$, $\mathbf{L}_2'\mathbf{U} = O_p(n)$, $\mathbf{L}_2'\mathbf{Z} = O_p(n)$, and $\mathbf{L}_1'\mathbf{Z} = O_p(n)$ then $\mathbf{L}_1'\mathbf{K}_1 = O_p(n)$, $\mathbf{K}_1'\mathbf{Z} = O_p(n)$, $\mathbf{K}_2'\mathbf{U} = O_p(n)$, $\mathbf{K}_1'\mathbf{K}_1 = O_p(n)$, and $\mathbf{K}_2'\mathbf{Z} = O_p(n)$. We have already shown that $\mathbf{L}_1'\mathbf{Z} = O_p(n)$ in (*ii*). We, therefore, focus on proving $\mathbf{L}_1'\mathbf{L}_1 = O_p(n)$, $\mathbf{L}_2'\mathbf{U} =$

$O_p(n)$, and $\mathbf{L}_2'\mathbf{Z} = O_p(n)$. Let $\mathbf{R}$ and $\mathbf{F}$ be generic notations for $\mathbf{L}_1$ or $\mathbf{L}_2$; and $\mathbf{L}_1$, $\mathbf{U}$, or $\mathbf{Z}$, respectively. For brevity, only $\mathbf{R}'\mathbf{F} = O_p(n)$ is proved and this follows in the same way by ergodicity, Cauchy-Schwarz and the moment conditions in Assumption 3 which ensure that $\mathbb{E}[\log^2(X_t)] < \infty$, $\mathbb{E}[\log^4(X_t)] < \infty$, $\mathbb{E}[U_t^2] < \infty$, and $\mathbb{E}[Z_{ti}^2] < \infty$.

(*iv*) From (*iii*), we note that the ergodic theorem applies to $n^{-1}\mathbf{L}_2'\mathbf{U}$ and $n^{-1}\mathbf{K}_2'\mathbf{U}$ and $\mathbb{E}[\log^2(X_t)U_t]$ $= 0$, so that $n^{-1}\mathbf{L}_2'\mathbf{U} = o_p(1)$ and $n^{-1}\mathbf{K}_2'\mathbf{U} = o_p(1)$, completing the proof. ■

Proof of Lemma A2: (*i*) We can obtain the first-order derivative with respect to γ as follows:

$$L_n^{(1)}(0;\alpha) = 2\mathbf{P}(\alpha)'\mathbf{Q}(0)[\mathbf{Q}(0)'\mathbf{Q}(0)]^{-1}\mathbf{K}_1\mathbf{P}(\alpha) + \mathbf{P}(\alpha)'\mathbf{Q}(0)(d/d\gamma)[\mathbf{Q}(0)'\mathbf{Q}(0)]^{-1}\mathbf{Q}(0)'\mathbf{P}(\alpha).$$

We also note that

$$(d/d\gamma)[\mathbf{Q}(0)'\mathbf{Q}(0)]^{-1} = -(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}, \tag{A.3}$$

and that $\mathbf{P}(\alpha) = \mathbf{Y} - \alpha\boldsymbol{\iota} = \mathbf{Z}[\alpha_* - \alpha, \xi_*]' + \mathbf{U} = \mathbf{Z}\boldsymbol{\kappa}(\alpha) + \mathbf{U}$ by letting that $\boldsymbol{\kappa}(\alpha) := [\alpha_* - \alpha, \xi_*]'$. Going forward we suppress α of $\boldsymbol{\kappa}(\alpha)$ for notational simplicity. It follows that

$$\begin{aligned} L_n^{(1)}(0;\alpha) = & \underbrace{2(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U})'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U})}_{(*)} \\ & \underbrace{-(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U})'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U})}_{(**)}. \end{aligned}$$

We now examine each component on the right side. The first component (*) can be expressed as a sum of four other components: (a) $2\boldsymbol{\kappa}'\mathbf{Z}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{Z}\boldsymbol{\kappa} = 2\boldsymbol{\kappa}'\mathbf{K}_1'\mathbf{Z}\boldsymbol{\kappa}$; (b) $2\boldsymbol{\kappa}'\mathbf{K}_1'\mathbf{U}$; (c) $2\mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{Z}\boldsymbol{\kappa} = 2\boldsymbol{\kappa}'\mathbf{Z}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$; and (d) $2\mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{U}$. Next, the second component (**) can also be expressed as a sum of four components: (a) $-\boldsymbol{\kappa}'\mathbf{Z}'\mathbf{K}_1\boldsymbol{\kappa} - \boldsymbol{\kappa}'\mathbf{K}_1'\mathbf{Z}\boldsymbol{\kappa} = -2\boldsymbol{\kappa}'\mathbf{K}_1'\mathbf{Z}\boldsymbol{\kappa}$; (b) $-\mathbf{U}'\mathbf{Z}(\mathbf{Z}'\,\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{K}_1\boldsymbol{\kappa} - \boldsymbol{\kappa}'\mathbf{K}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} = -2\boldsymbol{\kappa}'\mathbf{K}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$; (c) $-\mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{Z}\boldsymbol{\kappa} - \boldsymbol{\kappa}'\mathbf{Z}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} = -2\boldsymbol{\kappa}'\mathbf{Z}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$; and (d) $-\mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\,\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$. Adding and organizing all of these according to their orders of convergence yields the following

- (a) $2\boldsymbol{\kappa}'\mathbf{K}_1'\mathbf{Z}\boldsymbol{\kappa} - 2\boldsymbol{\kappa}'\mathbf{K}_1'\mathbf{Z}\boldsymbol{\kappa} = 0$;
- (b, c) $2\boldsymbol{\kappa}'\{\mathbf{K}_1' + \mathbf{Z}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' - \mathbf{K}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' - \mathbf{Z}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\}\mathbf{U} = 2(\alpha_* - \alpha)\mathbf{L}_1'\mathbf{M}\mathbf{U}$;
- (d) $2\mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{U} - \mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$,

so that the first-order derivative is now obtained as

$$L_n^{(1)}(0;\alpha) = 2(\alpha_* - \alpha)\mathbf{L}_1'\mathbf{M}\mathbf{U} + 2\mathbf{U}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} - \mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}.$$

(*ii*) Given the result in (*i*), we note that $\mathbf{L}_1'\mathbf{M}\mathbf{U} = \mathbf{L}_1'\mathbf{U} - \mathbf{L}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$, and Lemma A1(*i* and *ii*) implies that $\mathbf{L}_1'\mathbf{M}\mathbf{U} = O_p(\sqrt{n})$. We also note that $\mathbf{K}_1'\mathbf{U} = [\mathbf{L}_1'\mathbf{U} \vdots \mathbf{0}] = O_p(\sqrt{n})$, so that Lemma A1(*i* and *ii*) implies that $\mathbf{U}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} = O_p(1)$. Furthermore, Lemma A1(*i* and *ii*) implies that $\mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} = O_p(1)$. Therefore,

$$\begin{aligned} L_n^{(1)}(0;\alpha) &= 2(\alpha_* - \alpha)\underbrace{\mathbf{L}_1'\mathbf{M}\mathbf{U}}_{O_p(\sqrt{n})} + 2\underbrace{\mathbf{U}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}}_{O_p(1)} - \underbrace{\mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}}_{O_p(1)} \\ &= 2(\alpha_* - \alpha)\mathbf{L}_1'\mathbf{M}\mathbf{U} + o_p(\sqrt{n}). \end{aligned}$$

(*iii*) The second-order derivative is

$$\begin{aligned} L_n^{(2)}(0;\alpha) = {} & 2\mathbf{P}(\alpha)'\mathbf{K}_1[\mathbf{Q}(0)'\mathbf{Q}(0)]^{-1}\mathbf{K}_1'\mathbf{P}(\alpha) + 4\mathbf{P}(\alpha)'\mathbf{Q}(0)(d/d\gamma)[\mathbf{Q}(0)'\mathbf{Q}(0)]^{-1}\mathbf{K}_1'\mathbf{P}(\alpha) \\ & + 2\mathbf{P}(\alpha)'\mathbf{Q}(0)[\mathbf{Q}(0)'\mathbf{Q}(0)]^{-1}\mathbf{K}_2'\mathbf{P}(\alpha) + \mathbf{P}(\alpha)'\mathbf{Q}(0)(d^2/d\gamma^2)[\mathbf{Q}(0)'\mathbf{Q}(0)]^{-1}\mathbf{Q}(0)'\mathbf{P}(\alpha), \end{aligned}$$

where

$$\begin{aligned} (d^2/d\gamma^2)[\mathbf{Q}(0)'\mathbf{Q}(0)]^{-1} = {} & 2\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \\ & - (\mathbf{Z}'\mathbf{Z})^{-1}(2\mathbf{K}_1'\mathbf{K}_1 + \mathbf{Z}'\mathbf{K}_2 + \mathbf{K}_2'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}, \end{aligned} \quad \text{(A.4)}$$

and (A.3) already provides the specific form of $(d/d\gamma)[\mathbf{Q}(0)'\mathbf{Q}(0)]^{-1}$. Using these results and arranging them, we obtain the following second-order derivative:

$$\begin{aligned} L_n^{(2)}(0;\alpha) = {} & 2(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U})'\{\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1' + \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_2'\}(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U}) \\ & - 4(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U})'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U}) \\ & + 2(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U})'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1 + \mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U}) \\ & - (\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U})'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(2\mathbf{K}_1'\mathbf{K}_1 + \mathbf{Z}'\mathbf{K}_2 + \mathbf{K}_2'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Z}\boldsymbol{\kappa} + \mathbf{U}). \end{aligned} \quad \text{(A.5)}$$

We again organize this expression into three terms according to their orders:

- $2\boldsymbol{\kappa}'\{\mathbf{Z}'\mathbf{K}_1'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'+\mathbf{K}_2'\}\mathbf{Z}\boldsymbol{\kappa}-4\boldsymbol{\kappa}'(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{Z}\boldsymbol{\kappa}+2\boldsymbol{\kappa}'(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})\boldsymbol{\kappa}-\boldsymbol{\kappa}'(2\mathbf{K}_1'\mathbf{K}_1+\mathbf{Z}'\mathbf{K}_2+\mathbf{K}_2'\mathbf{Z})\boldsymbol{\kappa} = 2\boldsymbol{\kappa}'\mathbf{K}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{K}_1\boldsymbol{\kappa}-2\boldsymbol{\kappa}'\mathbf{K}_1'\mathbf{K}_1\boldsymbol{\kappa} = -2(\alpha_*-\alpha)^2\mathbf{L}_1'\mathbf{M}\mathbf{L}_1$;

- $4\boldsymbol{\kappa}'\mathbf{Z}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{U}-4\boldsymbol{\kappa}'(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{U}-4\boldsymbol{\kappa}'\mathbf{Z}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}+2\boldsymbol{\kappa}'\mathbf{K}_2'\mathbf{U}+2\boldsymbol{\kappa}'\mathbf{Z}'\mathbf{K}_2(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}+4\boldsymbol{\kappa}'(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} -2\boldsymbol{\kappa}'(2\mathbf{K}_1'\mathbf{K}_1+\mathbf{Z}'\mathbf{K}_2+\mathbf{K}_2'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} = 2(\alpha_*-\alpha)[\mathbf{L}_2'\mathbf{M}\mathbf{U}-2\mathbf{L}_1'\mathbf{M}\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}-2\mathbf{L}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{M}\mathbf{U}]$; and

- $2[\mathbf{U}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{U}+\mathbf{U}'\mathbf{K}_2(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}-2\mathbf{U}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}]+2\mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}[(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})-\mathbf{K}_1'\mathbf{K}_1-\mathbf{Z}'\mathbf{K}_2](\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$.

Next apply Lemma A1 to each term. First, the proof of Lemma A3 has already shown that $\mathbf{L}_1'\mathbf{M}\mathbf{L}_1 = O_p(n)$ and $\mathbf{L}_2'\mathbf{M}\mathbf{U} = o_p(n)$. Second, $\mathbf{L}_1'\mathbf{M}\mathbf{K}_1 = \mathbf{L}_1'\mathbf{K}_1 - \mathbf{L}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{K}_1$. Assumption 3 and Lemma A1(*ii*, *iii*, and *iv*) now imply that $\mathbf{L}_1'\mathbf{M}\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} = o_p(n)$. Third, $\mathbf{K}_1'\mathbf{M}\mathbf{U} = \mathbf{K}_1'\mathbf{U} - \mathbf{K}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} = o_p(n)$ by Lemma A1(*i* and *iv*), so that $\mathbf{L}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{M}\mathbf{U} = o_p(n)$ by Lemma A1(*ii* and *iii*). Therefore, $\mathbf{L}_2'\mathbf{M}\mathbf{U} - 2\mathbf{L}_1'\mathbf{M}\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} - 2\mathbf{L}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{M}\mathbf{U} = o_p(n)$. Finally, we combine all components in Lemma A1 and obtain that $\mathbf{U}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{K}_1'\mathbf{U} + \mathbf{U}'\mathbf{K}_2(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} - 2\mathbf{U}'\mathbf{K}_1(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} + \mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}[(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{K}_1+\mathbf{K}_1'\mathbf{Z}) - \mathbf{K}_1'\mathbf{K}_1 - \mathbf{Z}'\mathbf{K}_2](\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} = o_p(n)$. Thus, the first, third, and final facts now imply that $L_n^{(2)}(0;\alpha) = -2(\alpha_*-\alpha)^2\mathbf{L}_1'\mathbf{M}\mathbf{L}_1 + o_p(n)$. This completes the proof. ■

Proof of Lemma A3: (*i*) Applying a second-order Taylor expansion to $L_n(\gamma;\beta)$ and optimizing with respect to γ, we have

$$\sup_{\gamma}\{L_n(\gamma;\beta)-L_n(0;\beta)\} = -\frac{\{L_n^{(1)}(0;\beta)\}^2}{2L_n^{(2)}(0;\beta)} + o_p(1) = \frac{\{\beta\mathbf{L}_1'\mathbf{M}\mathbf{U}\}^2}{\beta^2\mathbf{L}_1'\mathbf{M}\mathbf{L}_1-\beta\mathbf{L}_2'\mathbf{M}\mathbf{U}} + o_p(1),$$

where $L_n^{(1)}(0;\beta) := (d/d\gamma)L_n^{(1)}(0;\beta) = 2\beta\mathbf{L}_1'\mathbf{M}\mathbf{U}$ and $L_n^{(2)}(0;\beta) := (d^2/d\gamma^2)L_n^{(1)}(0;\beta) = 2\beta\mathbf{L}_2'\mathbf{M}\mathbf{U} - 2\beta^2\mathbf{L}_1'\mathbf{M}\mathbf{L}_1$. In (*ii*), we show that $\mathbf{L}_2'\mathbf{M}\mathbf{U} = o_p(n)$, so that the desired result follows.

(*ii*) We partition the proof into three components. First, from the fact that $\mathbf{L}_1'\mathbf{M}\mathbf{U} = \mathbf{L}_1'\mathbf{U}-\mathbf{L}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$, Lemma A1(*i* and *ii*) and Assumption 3 imply that $\mathbf{L}_1'\mathbf{M}\mathbf{U} = O_p(\sqrt{n})$. Second, we note that $\mathbf{L}_1'\mathbf{M}\mathbf{L}_1 = \mathbf{L}_1'\mathbf{L}_1-\mathbf{L}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{L}_1$, so that Lemma A1(*ii* and *iii*) and Assumption 3 imply that $\mathbf{L}_1'\mathbf{M}\mathbf{L}_1 = O_p(n)$. Third, $\mathbf{L}_2'\mathbf{M}\mathbf{U} = \mathbf{L}_2'\mathbf{U} - \mathbf{L}_2'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$. Lemma A1(*ii* and *iii*) and Assumption 3 imply that

$\mathbf{L}_2'\mathbf{M}\mathbf{U} = O_p(n)$. Further, $\mathbf{L}_2'\mathbf{M}\mathbf{U} = \mathbf{L}_2'\mathbf{U} - \mathbf{L}_2'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$. Thus, $\mathbf{L}_2'\mathbf{M}\mathbf{U} = o_p(n)$ by Lemma A1(*iv*). Given these results, it now follows that the right side of (3) is $O_p(1)$ as desired. ■

Proof of Lemma A4: (*i*) Applying a second-order Taylor expansion to $L_n(\gamma;\alpha)$ and optimizing with respect to γ, we have

$$\sup_{\gamma}\{L_n(\gamma;\alpha) - L_n(0;\alpha)\} = -\frac{\{L_n^{(1)}(0;\alpha)\}^2}{2L_n^{(2)}(0;\alpha)} + o_p(1) = \frac{\{2(\alpha_* - \alpha)n^{-1/2}\mathbf{L}_1'\mathbf{M}\mathbf{U}\}^2}{4(\alpha_* - \alpha)^2 n^{-1}\mathbf{L}_1'\mathbf{M}\mathbf{L}_1} + o_p(1)$$

using Lemma A2(*ii* and *iii*), so that the desired result follows.

(*ii*) This is obvious from Lemmas A3 and A4(*i*). ■

Proof of Theorem 2: The desired results immediately follow from Lemmas A3 and A4. In particular, we applied the MDS (martingale difference sequence) CLT and the continuous mapping theorem to derive the asymptotic null distribution of $\mathcal{Z}_0$. ■

Before proving Theorem 3, we provide supplementary lemmas to assist in an efficient proof.

Lemma A5. *Given Assumptions 1 and 4,*

(i) $\mathbf{C}_1'\mathbf{U} = O_p(\sqrt{n})$, $\mathbf{Z}'\mathbf{U} = O_p(\sqrt{n})$, $\mathbf{J}_1'\mathbf{U} = O_p(\sqrt{n})$, *where for* $j = 1, 2, \ldots$, $\mathbf{J}_j := [\mathbf{0}_{n\times 1} \vdots \mathbf{C}_j \vdots \mathbf{0}_{n\times k}]$*;*

(ii) $\mathbf{C}_1'\mathbf{Z} = O_p(n)$, $\mathbf{Z}'\mathbf{Z} = O_p(n)$, $\mathbf{J}_1'\mathbf{Z} = O_p(n)$*;*

(iii) $\mathbf{C}_1'\mathbf{C}_1 = O_p(n)$, $\mathbf{C}_1'\mathbf{J}_1 = O_p(n)$, $\mathbf{C}_2'\mathbf{U} = O_p(n)$, $\mathbf{C}_2'\mathbf{Z} = O_p(n)$, $\mathbf{J}_1'\mathbf{Z} = O_p(n)$, $\mathbf{J}_1'\mathbf{J}_1 = O_p(n)$, $\mathbf{J}_2'\mathbf{U} = O_p(n)$, *and* $\mathbf{J}_2'\mathbf{Z} = O_p(n)$*; and*

(iv) $\mathbf{C}_2'\mathbf{U} = o_p(n)$ *and* $\mathbf{J}_2'\mathbf{U} = o_p(n)$. □

Lemma A6. *Given Assumptions 1, 4, and* $\mathcal{H}_0'''$*,*

(i) $L_n^{(1)}(1;\xi) = 2(\xi_* - \xi)\mathbf{C}_1'\mathbf{M}\mathbf{U} + 2\mathbf{U}'\mathbf{J}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U} - \mathbf{U}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{J}_1 + \mathbf{J}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$, *where for* $j = 1, 2, \ldots$, $L_n^{(j)}(1;\xi) := (\partial^j/\partial\gamma^j)L_n(\gamma;\xi)|_{\gamma=1}$*;*

(ii) $L_n^{(1)}(1;\xi) = 2(\xi_* - \xi)\mathbf{C}_1'\mathbf{M}\mathbf{U} + o_p(\sqrt{n})$*; and*

(iii) $L_n^{(2)}(1;\xi) = -2(\xi_* - \xi)^2\mathbf{C}_1'\mathbf{M}\mathbf{C}_1 + o_p(n)$. □

Lemma A7. *Given Assumptions 1, 4, and* $\mathcal{H}_0'''$*,*

(i) $QLR_n^{(\gamma=1;\beta)} = \{\mathbf{C}_1'\mathbf{M}\mathbf{U}\}^2/\{\hat{\sigma}_{n,0}^2(\mathbf{C}_1'\mathbf{M}\mathbf{C}_1)\} + o_p(1)$*; and*

(ii) $QLR_n^{(\gamma=1;\beta)} = O_p(1)$. □

Lemma A8. *Given Assumptions 1, 4, and* $\mathcal{H}_0'''$*,* *(i)* $QLR_n^{(\gamma=1;\xi)} = \{\mathbf{C}_1'\mathbf{MU}\}^2/\{\hat{\sigma}_{n,0}^2(\mathbf{C}_1'\mathbf{MC}_1)\} + o_p(1)$*; and*

(ii) $QLR_n^{(\gamma=1;\xi)} = O_p(1)$. □

Proof of Lemma A5: (*i*) The plan of this proof is similar to that of Lemma A1. By the definition of $\mathbf{J}_1$, we note that if $\mathbf{C}_1'\mathbf{U} = O_p(\sqrt{n})$, $\mathbf{J}_1'\mathbf{U} = O_p(\sqrt{n})$. We also note that the moment condition in Assumption 4 is stronger than that of Assumption 3. This implies that $\mathbf{Z}'\mathbf{U} = O_p(\sqrt{n})$ follows from Lemma A1(*i*). We therefore focus on proving $\mathbf{C}_1'\mathbf{U} = O_p(\sqrt{n})$.

From the definition of $\mathbf{C}_1'\mathbf{U}$, we note that $n^{-1/2}\mathbf{C}_1'\mathbf{U} = n^{-1/2}\sum_{t=1}^n X_t \log(X_t)U_t$, and we can apply the CLT if $\mathbb{E}[X_t^2\log^2(X_t)U_t^2] < \infty$. Note that $\mathbb{E}[X_t^2\log^2(X_t)U_t^2] \leq \mathbb{E}[X_t^4\log^4(X_t)]^{1/2}\ \mathbb{E}[U_t^4]^{1/2} \leq \mathbb{E}[X_t^8]^{1/4}\mathbb{E}[\log^8(X_t)]^{1/4}\mathbb{E}[U_t^4]^{1/2}$ by applying Cauchy-Schwarz. Each element in the right side is finite by Assumption 4(*ii.a*), so that $\mathbb{E}[X_t^2\log^2(X_t)U_t^2] < \infty$. Alternatively, $\mathbb{E}[X_t^2\log^2(X_t)U_t^2] \leq \mathbb{E}[X_t^4]^{1/2}\mathbb{E}[\log^4 (X_t)U_t^4]^{1/2} \leq \mathbb{E}[X_t^4]^{1/2}\mathbb{E}[\log^8(X_t)]^{1/4}\mathbb{E}[U_t^8]^{1/4}$, and Assumption 4(*ii.b*) implies that the right side is finite. Finally, we note that $\mathbb{E}[X_t^2\log^2(X_t)U_t^2] \leq \mathbb{E}[\log^4(X_t)]^{1/2}\mathbb{E}[X_t^4U_t^4]^{1/2} \leq \mathbb{E}[\log^4(X_t)]^{1/2}\ \mathbb{E}[X_t^8]^{1/4}\mathbb{E}[U_t^8]^{1/4}$, and Assumption 4(*ii.c*) implies that the right side is finite. Thus, $\mathbf{C}_1'\mathbf{U} = O_p(\sqrt{n})$.

(*ii*) As in (*i*), if $\mathbf{C}_1'\mathbf{Z} = O_p(n)$, $\mathbf{J}_1'\mathbf{Z} = O_p(n)$ by the definition of $\mathbf{J}_1$. Furthermore, Lemma A1(*ii*) already shows that $\mathbf{Z}'\mathbf{Z} = O_p(n)$, and the current moment condition is stronger than Assumption 3, so that $\mathbf{Z}'\mathbf{Z} = O_p(n)$. We therefore focus on proving $\mathbf{C}_1'\mathbf{Z} = O_p(n)$. By definition $n^{-1}\mathbf{C}_1'\mathbf{Z} = [n^{-1}\sum X_t\log(X_t) W_{t,j}]$, so that if $\mathbb{E}[|X_t\log(X_t)W_{t,j}|] < \infty$, the egodict theorem holds, giving the desired result. We first consider the case where $X_t = W_{t,j}$. If so, $\mathbb{E}[|X_t\log(X_t)W_{t,j}|] = \mathbb{E}[|X_t^2\log(X_t)|] \leq \mathbb{E}[X_t^4]^{1/2}\mathbb{E}[\log^2(X_t)]^{1/2} < \infty$ by Cauchy-Schwarz and Assumption 4. Next consider the case where $X_t \neq W_{t,j}$: (*a*) $\mathbb{E}[|X_t\log (X_t)W_{t,j}|] \leq \mathbb{E}[|X_t\log(X_t)|^2]^{1/2}\mathbb{E}[W_{t,j}^2]^{1/2} \leq \mathbb{E}[X_t^4]^{1/4}\mathbb{E}[|\log^4(X_t)]^{1/4}\mathbb{E}[W_{t,j}^2]^{1/2}$; (*b*) $\mathbb{E}[|X_t\log(X_t) W_{t,j}|] \leq \mathbb{E}[|X_tW_{t,j}|^2]^{1/2}\mathbb{E}[\log^2(X_t)]^{1/2} \leq \mathbb{E}[X_t^4]^{1/4}\mathbb{E}[W_{t,j}^4]^{1/4}\mathbb{E}[\log^2(X_t)]^{1/2}$; and finally (*c*) $\mathbb{E}[|X_t\log (X_t)W_{t,j}|] \leq \mathbb{E}[|\log(X_t)W_{t,j}|^2]^{1/2}\mathbb{E}[X_t^2]^{1/2} \leq \mathbb{E}[\log^4(X_t)]^{1/4}\mathbb{E}[W_{t,j}^4]^{1/4}\mathbb{E}[X_t^2]^{1/2}$ by Cauchy-Schwarz. Note that the elements on the right side of (*a*), (*b*), and (*c*) are finite by Assumption 4.

(*iii*) By the definition of $\mathbf{J}_1$ and $\mathbf{J}_2$, if $\mathbf{C}_1'\mathbf{C}_1 = O_p(n)$, $\mathbf{C}_2'\mathbf{U} = O_p(n)$, $\mathbf{C}_2'\mathbf{Z} = O_p(n)$, and $\mathbf{C}_1'\mathbf{Z} = O_p(n)$, then $\mathbf{C}_1'\mathbf{J}_1 = O_p(n)$, $\mathbf{J}_1'\mathbf{Z} = O_p(n)$, $\mathbf{J}_2'\mathbf{U} = O_p(n)$, $\mathbf{J}_1'\mathbf{J}_1 = O_p(n)$, and $\mathbf{J}_2'\mathbf{Z} = O_p(n)$. We have already shown that $\mathbf{C}_1'\mathbf{Z} = O_p(n)$ in (*ii*). We therefore focus on proving $\mathbf{C}_1'\mathbf{C}_1 = O_p(n)$, $\mathbf{C}_2'\mathbf{U} = O_p(n)$, and $\mathbf{C}_2'\mathbf{Z} = O_p(n)$.

We examine each case in turn. (*a*) Note that $n^{-1}\mathbf{C}_1'\mathbf{C}_1 = n^{-1}\sum X_t^2\log^2(X_t)$, so that if $\mathbb{E}[X_t^2\log^2(X_t)] < \infty$, the ergodic theorem holds. We also note that $\mathbb{E}[X_t^2\log^2(X_t)] \leq \mathbb{E}[X_t^4]^{1/2}\mathbb{E}[\log^4(X_t)]^{1/2}$, and the right side is finite by Assumption 4. (*b*) Note that $n^{-1}\mathbf{C}_2'\mathbf{U} = n^{-1}\sum X_t\log^2(X_t)U_t$ and the ergodic theorem holds if $\mathbb{E}[|X_t \log^2(X_t)U_t|] < \infty$. Furthermore, we note that (*b.i*) $\mathbb{E}[|X_t \log^2(X_t)U_t|] \leq \mathbb{E}[|X_t \log^2(X_t)|^2]^{1/2}\mathbb{E}[U_t^2]^{1/2} \leq \mathbb{E}[X_t^4]^{1/4}\mathbb{E}[\log^8(X_t)]^{1/4}\mathbb{E}[U_t^2]^{1/2}$; (*b.ii*) $\mathbb{E}[|X_t \log^2(X_t)U_t|] \leq \mathbb{E}[|U_t \log^2 (X_t)|^2]^{1/2}\mathbb{E}[X_t^2]^{1/2} \leq \mathbb{E}[U_t^4]^{1/4}\mathbb{E}[\log^8(X_t)]^{1/4}\mathbb{E}[X_t^2]^{1/2}$; and finally (*b.iii*) $\mathbb{E}[|X_t \log^2(X_t)U_t|] \leq \mathbb{E}[|U_t X_t|^2]^{1/2}\mathbb{E}[\log^4(X_t)]^{1/2} \leq \mathbb{E}[|U_t|^4]^{1/4}\mathbb{E}[X_t^4]^{1/4}\mathbb{E}[\log^4(X_t)]^{1/2}$. We further note that each element forming the right sides of these upper bounds is finite by Assumption 4(*ii.a*), 4(*ii.b*), and 4(*ii.c*), respectively. Thus, $\mathbb{E}[|X_t \log^2(X_t)U_t|] < \infty$. (*c*) Finally, we note that $n^{-1}\mathbf{C}_2'\mathbf{Z} = [n^{-1}\sum X_t \log^2 (X_t)W_{t,j}]$, so that if $\mathbb{E}[|X_t \log^2(X_t)W_{t,j}|] < \infty$, the ergodic theorem applies. First, if $W_{t,j} = X_t$, the proof is the same as that for $\mathbb{E}[X_t^2 \log^2(X_t)] < \infty$, which we have just proved. Second, if $W_{t,j} \neq X_t$, by the same argument as in (*b*), (*c.i*) $\mathbb{E}[|X_t \log^2(X_t)W_{t,j}|] \leq \mathbb{E}[X_t^4]^{1/4}\mathbb{E}[\log^8(X_t)]^{1/4} \mathbb{E}[W_{t,j}^2]^{1/2}$; (*c.ii*) $\mathbb{E}[|X_t \log^2(X_t)W_{t,j}|] \leq \mathbb{E}[W_{t,j}^4]^{1/4}\mathbb{E}[\log^8(X_t)]^{1/4}\mathbb{E}[X_t^2]^{1/2}$; and (*c.iii*) $\mathbb{E}[|X_t \log^2(X_t)W_{t,j}|] \leq \mathbb{E}[W_{t,j}^4]^{1/4}\mathbb{E}[X_t^4]^{1/4}\mathbb{E}[\log^4(X_t)]^{1/2}$. Given these, the right sides in (*c.i*), (*c.ii*), and (*c.iii*) are finite if Assumption 4(*ii.a*), 4(*ii.b*) or 4(*ii.c*) holds. Thus, $\mathbb{E}[|X_t \log^2(X_t)W_{t,j}|] < \infty$.

(*iv*) From the proof of (*iii*), the ergodic theorem applies to $n^{-1}\mathbf{C}_2'\mathbf{U}$ and $n^{-1}\mathbf{J}_2'\mathbf{U}$. Furthermore, $\mathbb{E}[X_t \log^2(X_t)U_t] = 0$, so that $n^{-1}\mathbf{C}_2'\mathbf{U} = o_p(1)$ and $n^{-1}\mathbf{J}_2'\mathbf{U} = o_p(1)$. This completes the proof. ■

Proof of Lemma A6: (*i*) The first-order derivative with respect to γ is

$$\frac{\partial}{\partial\gamma}L_n(\gamma;\xi) = 2\widetilde{\mathbf{P}}(\xi)'\widetilde{\mathbf{Q}}(\gamma)[\widetilde{\mathbf{Q}}(\gamma)'\widetilde{\mathbf{Q}}(\gamma)]^{-1}\frac{\partial}{\partial\gamma}\widetilde{\mathbf{Q}}(\gamma)'\widetilde{\mathbf{P}}(\xi) + \widetilde{\mathbf{P}}(\xi)'\widetilde{\mathbf{Q}}(\gamma)\frac{\partial}{\partial\gamma}[\widetilde{\mathbf{Q}}(\gamma)'\widetilde{\mathbf{Q}}(\gamma)]^{-1}\widetilde{\mathbf{Q}}(\gamma)'\widetilde{\mathbf{P}}(\xi).$$

When $\gamma = 1$, we can write the derivative as follows:

$$L_n^{(1)}(1;\xi) = 2\widetilde{\mathbf{P}}(\xi)'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{J}_1'\widetilde{\mathbf{P}}(\xi) + \widetilde{\mathbf{P}}(\xi)'\mathbf{Z}(d/d\gamma)[\widetilde{\mathbf{Q}}(1)'\widetilde{\mathbf{Q}}(1)]^{-1}\mathbf{Z}'\widetilde{\mathbf{P}}(\xi).$$

We also note that

$$(d/d\gamma)[\widetilde{\mathbf{Q}}(1)'\widetilde{\mathbf{Q}}(1)]^{-1} = -(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{J}_1 + \mathbf{J}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1} \quad \text{(A.6)}$$

and that $\widetilde{\mathbf{P}}(\xi) = (\mathbf{Y} - \xi\mathbf{X}) = \mathbf{Z}[\alpha_*, \xi_* - \xi, \boldsymbol{\eta}_*']' + \mathbf{Z}\mathbf{U} = \mathbf{Z}\boldsymbol{\zeta}(\xi) + \mathbf{U}$ by letting $\boldsymbol{\zeta}(\xi) := [\alpha_*, \xi_* - \xi, \boldsymbol{\eta}_*']'$. Going forward, we suppress ξ in $\boldsymbol{\zeta}(\xi)$ for notational simplicity. Then, it follows that

$$L_n^{(1)}(1;\xi) = 2(\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U})'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{J}_1'(\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U}) - (\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U})'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{J}_1+\mathbf{J}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U}).$$

(*ii*) We note that the form of $L_n^{(1)}(1;\xi)$ is identical to the form of $L_n^{(1)}(0;\alpha)$ in Lemma A2(*i*), provided that $(\xi_* - \xi)$, $\mathbf{C}_1$, and $\mathbf{J}_1$ are replaced by $(\alpha_* - \alpha)$, $\mathbf{L}_1$, and $\mathbf{K}_1$, respectively. Furthermore, the contents of

Lemma A5 are also identical to those of Lemma A1, provided that $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{J}_1$, and $\mathbf{J}_2$ are replaced by $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{K}_1$, and $\mathbf{K}_2$, respectively. Thus, we can repeat the proof of Lemma A2(*ii*) for the proof here because Lemma A2(*ii*) holds as a corollary of Lemma A1.

(*iii*) We now examine the second-order derivative. We obtain

$$\begin{aligned} L_n^{(2)}(1;\xi) =& 2\widetilde{\mathbf{P}}(\xi)'\mathbf{J}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{J}_1'\widetilde{\mathbf{P}}(\xi) + 4\widetilde{\mathbf{P}}(\xi)'\mathbf{Z}(d/d\gamma)[\widetilde{\mathbf{Q}}(1)'\widetilde{\mathbf{Q}}(1)]^{-1}\mathbf{J}_1'\widetilde{\mathbf{P}}(\xi) \\ &+ 2\widetilde{\mathbf{P}}(\xi)'\mathbf{Z}(d/d\gamma)[\widetilde{\mathbf{Q}}(1)'\widetilde{\mathbf{Q}}(1)]^{-1}\mathbf{J}_1'\widetilde{\mathbf{P}}(\xi) + \widetilde{\mathbf{P}}(\xi)'\mathbf{Z}(d^2/d\gamma^2)[\widetilde{\mathbf{Q}}(1)'\widetilde{\mathbf{Q}}(1)]^{-1}\mathbf{Z}'\widetilde{\mathbf{P}}(\xi), \end{aligned}$$

where $(d^2/d\gamma^2)[\widetilde{\mathbf{Q}}(1)'\widetilde{\mathbf{Q}}(1)]^{-1} = 2(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{J}_1+\mathbf{J}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{J}_1+\mathbf{J}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1} - (\mathbf{Z}'\mathbf{Z})^{-1}(2\mathbf{J}_1'\mathbf{J}_1 +\mathbf{Z}'\mathbf{J}_2+\mathbf{J}_2'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}$, and (A.6) already provides the form of $(d/d\gamma)[\widetilde{\mathbf{Q}}(1)'\widetilde{\mathbf{Q}}(1)]^{-1}$. Using these expressions and rearranging, we obtain the following second-order derivative:

$$\begin{aligned} L_n^{(2)}(1;\xi) = & 2(\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U})'\{\mathbf{J}_1(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{J}_1' + \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{J}_2'\}(\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U}) \\ & - 4(\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U})'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{J}_1+\mathbf{J}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{J}_1'(\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U}) \\ & + 2(\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U})'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{J}_1+\mathbf{J}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{J}_1+\mathbf{J}_1'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U}) \\ & - (\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U})'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}(2\mathbf{J}_1'\mathbf{J}_1+\mathbf{Z}'\mathbf{J}_2+\mathbf{J}_2'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Z}\boldsymbol{\zeta}+\mathbf{U}). \end{aligned}$$

We again note that the form of $L_n^{(2)}(1;\xi)$ is identical to that of $L_n^{(2)}(0;\alpha)$ in (A.5), provided that $\mathbf{J}_1$, $\mathbf{J}_2$, and $\boldsymbol{\zeta}$ are replaced by $\mathbf{K}_1$, $\mathbf{K}_2$, and $\boldsymbol{\kappa}$, respectively. Given Lemma A5, we may again repeat the proof of Lemma A2(*iii*) for the proof here as in the proof of (*ii*). ■

Proof of Lemma A7: (*i*) Applying a second-order Taylor expansion to $L_n(\gamma;\beta)$ and optimizing with respect to γ, we have

$$\sup_{\gamma}\{L_n(\gamma;\beta) - L_n(1;\beta)\} = -\frac{\{L_n^{(1)}(1;\beta)\}^2}{2L_n^{(2)}(1;\beta)} + o_p(1) = \frac{\{\beta\mathbf{C}_1'\mathbf{M}\mathbf{U}\}^2}{\beta^2\mathbf{C}_1'\mathbf{M}\mathbf{C}_1 - \beta\mathbf{C}_2'\mathbf{M}\mathbf{U}} + o_p(1),$$

In (*ii*), we show that $\mathbf{C}_2'\mathbf{M}\mathbf{U} = o_p(n)$, so that the desired result follows.

(*ii*) We partition the proof into three components. First, from the fact that $\mathbf{C}_1'\mathbf{M}\mathbf{U} = \mathbf{C}_1'\mathbf{U} - \mathbf{C}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$, Lemma A5(*i* and *ii*) and Assumption 4 imply that $\mathbf{C}_1'\mathbf{M}\mathbf{U} = O_p(\sqrt{n})$. Second, we note that $\mathbf{C}_1'\mathbf{M}\mathbf{C}_1 = \mathbf{C}_1'\mathbf{C}_1 - \mathbf{C}_1'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{C}_1$, so that Lemma A5(*ii* and *iii*) and Assumption 4 imply that $\mathbf{C}_1'\mathbf{M}\mathbf{C}_1 = O_p(n)$. Third, $\mathbf{C}_2'\mathbf{M}\mathbf{U} = \mathbf{C}_2'\mathbf{U} - \mathbf{C}_2'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$. Lemma A5(*ii* and *iii*) and Assumption 4 imply that $\mathbf{C}_2'\mathbf{M}\mathbf{U} = O_p(n)$. Further, $\mathbf{C}_2'\mathbf{M}\mathbf{U} = \mathbf{C}_2'\mathbf{U} - \mathbf{C}_2'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{U}$. Thus, $\mathbf{C}_2'\mathbf{M}\mathbf{U} = o_p(n)$

by Lemma A5(*iv*). Given these findings, the desired result now follows. ■

Proof of Lemma A8: (*i*) Applying a second-order Taylor expansion to $L_n(\gamma;\xi)$ and optimizing with respect to γ, we have

$$\sup_{\gamma}\{L_n(\gamma;\xi)-L_n(1;\xi)\} = -\frac{\{L_n^{(1)}(1;\xi)\}^2}{2L_n^{(2)}(1;\xi)}+o_p(1) = \frac{\{2(\xi_*-\xi)n^{-1/2}\mathbf{C}_1'\mathbf{M}\mathbf{U}\}^2}{4(\xi_*-\xi)^2 n^{-1}\mathbf{C}_1'\mathbf{M}\mathbf{C}_1}+o_p(1)$$

using Lemma A6(*ii* and *iii*), so that the desired result follows.

(*ii*) This is obvious from Lemmas A7 and A8(*i*). ■

Proof of Theorem 3: The desired results immediately follow from Lemmas A7 and A8. In particular, we applied the MDS CLT and the continuous mapping theorem to derive the asymptotic null distribution of $\mathcal{Z}_1$. ■

Proof of Lemma 1: (*i*) We examine the second-order derivative of $N_n(\gamma)$ and $D_n(\gamma)$ and let γ converge to zero. That is,

$$\operatorname*{plim}_{\gamma\to 0} N_n^{(2)}(\gamma) = \operatorname*{plim}_{\gamma\to 0} 2\left\{(d/d\gamma)\mathbf{X}(\gamma)'\mathbf{M}\mathbf{U}\right\}^2 + 2\{\mathbf{X}(\gamma)'\mathbf{M}\mathbf{U}\}\left\{(d^2/d\gamma^2)\mathbf{X}(\gamma)'\mathbf{M}\mathbf{U}\right\} = 2\{\mathbf{L}_1\mathbf{M}\mathbf{U}\}^2$$

because $\operatorname{plim}_{\gamma\to 0}(d/d\gamma)\mathbf{X}(\gamma)=\mathbf{L}_1$ and $\operatorname{plim}_{\gamma\to 0}\mathbf{X}(\gamma)'\mathbf{M}\mathbf{U}=\boldsymbol{\iota}'\mathbf{M}\mathbf{U}=\mathbf{0}$. We further note that

$$\operatorname*{plim}_{\gamma\to 0}(d^2/d\gamma^2)D_n(\gamma) = \operatorname*{plim}_{\gamma\to 0} 2\left\{(d^2/d\gamma^2)\mathbf{X}(\gamma)'\mathbf{M}\mathbf{X}(\gamma) + (d/d\gamma)\mathbf{X}(\gamma)'\mathbf{M}(d/d\gamma)\mathbf{X}(\gamma)\right\} = 2\mathbf{L}_1\mathbf{M}\mathbf{L}_1$$

because $\operatorname{plim}_{\gamma\to 0}(d^2/d\gamma^2)\mathbf{X}(\gamma)'\mathbf{M}\mathbf{X}(\gamma)=\mathbf{L}_2'\mathbf{M}\boldsymbol{\iota}=\mathbf{0}$ and $\operatorname{plim}_{\gamma\to 0}(d/d\gamma)\mathbf{X}(\gamma)=\mathbf{L}_1$.

(*ii*) We now examine the second-order derivative of $N_n(\gamma)$ and $D_n(\gamma)$ and let γ converge to one. That is,

$$\operatorname*{plim}_{\gamma\to 1} N_n^{(2)}(\gamma) = \operatorname*{plim}_{\gamma\to 1} 2\left\{(d/d\gamma)\mathbf{X}(\gamma)'\mathbf{M}\mathbf{U}\right\}^2 + 2\{\mathbf{X}(\gamma)'\mathbf{M}\mathbf{U}\}\left\{(d^2/d\gamma^2)\mathbf{X}(\gamma)'\mathbf{M}\mathbf{U}\right\} = 2\{\mathbf{C}_1\mathbf{M}\mathbf{U}\}^2$$

because $\operatorname{plim}_{\gamma\to 1}(d/d\gamma)\mathbf{X}(\gamma)=\mathbf{C}_1$ and $\operatorname{plim}_{\gamma\to 1}\mathbf{X}(\gamma)'\mathbf{M}\mathbf{U}=\mathbf{X}'\mathbf{M}\mathbf{U}=\mathbf{0}$. We also note that

$$\operatorname*{plim}_{\gamma\to 1} D_n^{(2)}(\gamma) = \operatorname*{plim}_{\gamma\to 1} 2\left\{(d^2/d\gamma^2)\mathbf{X}(\gamma)'\mathbf{M}\mathbf{X}(\gamma) + (d/d\gamma)\mathbf{X}(\gamma)'\mathbf{M}(d/d\gamma)\mathbf{X}(\gamma)\right\} = 2\mathbf{C}_1\mathbf{M}\mathbf{C}_1,$$

from the fact that $\operatorname{plim}_{\gamma\to 1}(d^2/d\gamma^2)\mathbf{X}(\gamma)'\mathbf{M}\mathbf{X}(\gamma)=\mathbf{C}_2'\mathbf{M}\mathbf{X}=\mathbf{0}$ and $\operatorname{plim}_{\gamma\to 1}(d/d\gamma)\mathbf{X}(\gamma)=\mathbf{C}_1$. ■

Proof of Theorem 4: The results in Lemma 1 imply that

$$\sup_{\gamma \in \mathbf{\Gamma}} \frac{\{\mathbf{X}(\gamma)'\mathbf{MU}\}^2}{\hat{\sigma}_{n,0}^2 \mathbf{X}(\gamma)'\mathbf{MX}(\gamma)} \geq \max \left[\frac{\{\mathbf{L}_1 \mathbf{MU}\}^2}{\hat{\sigma}_{n,0}^2 \mathbf{L}_1 \mathbf{ML}_1}, \frac{\{\mathbf{C}_1 \mathbf{MU}\}^2}{\hat{\sigma}_{n,0}^2 \mathbf{C}_1 \mathbf{MC}_1} \right].$$

Therefore, the desired results hold as corollaries of Theorems 1, 2, and 3. ■

Proof of Theorem 5: We first note that $\mathbb{P}(\mathbb{E}[V_t|\mathbf{W}_t, X_t] = 0) < 1$ under the given condition, implying that $\mathbb{P}(\mathbb{E}[V_t|\mathbf{W}_t, \log(X_t)] = 0) < 1$ because $\log(\cdot)$ is a one-to-one mapping. Thus, theorem 2 of Bierens (1982) implies that for some $j_* \in \mathbb{N}$,

$$\mathbb{E}[V_t \log^{j_*}(X_t)] \neq 0. \tag{A.7}$$

We next consider $\min_{\boldsymbol{\delta},\beta} \mathbb{E}[(Y_t - \mathbf{Z}_t'\boldsymbol{\delta} - \beta X_t^{\gamma})^2]$. Note that $\mathbb{E}[(Y_t - \mathbf{Z}_t'\boldsymbol{\delta} - \beta X_t^{\gamma})^2] = \mathbb{E}[(U_t + \mathbb{E}[Y_t|\mathbf{W}_t] - \mathbf{Z}_t'\boldsymbol{\delta} - \beta X_t^{\gamma})^2] = \mathbb{E}[U_t^2] + \mathbb{E}[(\mathbb{E}[Y_t|\mathbf{W}_t] - \mathbf{Z}_t'\boldsymbol{\delta} - \beta X_t^{\gamma})^2]$, so that it follows that $\min_{\boldsymbol{\delta},\beta} \mathbb{E}[(Y_t - \mathbf{Z}_t'\boldsymbol{\delta} - \beta X_t^{\gamma})^2] = \mathbb{E}[U_t^2] + \mathbb{E}[(\mathbf{Z}_t'(\boldsymbol{\delta}_* - \widetilde{\boldsymbol{\delta}}) + (m(X_t) - \widetilde{\beta} X_t^{\gamma}))^2]$ by noting that $\mathbb{E}[Y_t|\mathbf{W}_t] = \mathbf{Z}_t'\boldsymbol{\delta}_* + m(X_t)$, where

$$\begin{bmatrix} \widetilde{\boldsymbol{\delta}} \\ \widetilde{\beta} \end{bmatrix} := \begin{bmatrix} \mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t'] & \mathbb{E}[\mathbf{Z}_t X_t^{\gamma}] \\ \mathbb{E}[\mathbf{Z}_t' X_t^{\gamma}] & \mathbb{E}[X_t^{2\gamma}] \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{E}[\mathbf{Z}_t(\mathbf{Z}_t'\boldsymbol{\delta}_* + m(X_t))] \\ \mathbb{E}[X_t^{\gamma}(\mathbf{Z}_t'\boldsymbol{\delta}_* + m(X_t))] \end{bmatrix}.$$

From this, it now follows that $\min_{\boldsymbol{\delta},\beta} \mathbb{E}[(Y_t - \mathbf{Z}_t'\boldsymbol{\delta} - \beta X_t^{\gamma})^2] = h(\gamma)$, where

$$h(\gamma) := \mathbb{E}[U_t^2] + \mathrm{var}[Q_t](1 - \mathrm{cov}[Q_t, U_t(\gamma)]^2/\{\mathrm{var}[U_t(\gamma)]\mathrm{var}[Q_t]\}),$$

$$Q_t := m(X_t) - \mathbf{Z}_t'\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbb{E}[\mathbf{Z}_t m(X_t)].$$

Note that we can apply the ULLN, so that $QLR_n/n = \sup_{\gamma \in \mathbf{\Gamma}}(1 - h(\gamma)/h_0) + o_p(1)$. Here,

$$1 - \frac{h(\gamma)}{h_0} = \left(\frac{\mathrm{var}[Q_t]}{\mathrm{var}[U_t] + \mathrm{var}[Q_t]} \right) \mathrm{corr}^2[U_t(\gamma), Q_t]$$

because $h_0 = \mathrm{var}[U_t] + \mathrm{var}[Q_t]$. Note that $\mathrm{var}[Q_t]/(\mathrm{var}[U_t] + \mathrm{var}[Q_t]) \in (0, 1)$ and $\mathrm{corr}^2[U_t(\cdot), Q_t] \in [0, 1)$ if there is no (β_*, γ_*) such that $m(X_t) = \beta_* X_t^{\gamma_*}$ w.p. 1: there is no c such that for each γ, $U_t(\gamma) = c \cdot Q_t$. If there is such a (β_*, γ_*), $\mathrm{corr}^2[U_t(\gamma_*), Q_t] = 1$, so that $QLR_n/n = \mathrm{var}[Q_t]/(\mathrm{var}[U_t] + \mathrm{var}[Q_t]) + o_p(1)$, and the proof is trivially completed.

We therefore from now suppose that there is no (β_*, γ_*) such that $m(X_t) = \beta_* X_t^{\gamma_*}$ w.p. 1. The desired proof is completed by showing that there is at least a single $\gamma \in \mathbf{\Gamma}$ such that $\mathrm{corr}^2[U_t(\gamma), Q_t] > 0$. In other words, if we show that $g(\cdot) := \mathrm{corr}[U_t(\cdot), Q_t]$ is not a zero-function, the proof is completed. To show this by contradiction, we suppose that $g(\cdot)$ is constant on $\mathbf{\Gamma}$, so that for each $\gamma \in \mathbf{\Gamma}$,

$$g'(\gamma) = \frac{1}{\text{var}^{3/2}[U_t(\gamma)]\text{var}[Q_t]}(\text{cov}[W_t(\gamma), Q_t]\text{var}[U_t(\gamma)] - \text{cov}[U_t(\gamma), Q_t]\text{cov}[W_t(\gamma), U_t(\gamma)]) = 0,$$

where $W_t(\gamma) := X_t^\gamma \log(X_t) - \mathbf{Z}_t'\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbb{E}[\mathbf{Z}_t X_t^\gamma \log(X_t)]$, and this implies that

$$\text{corr}[W_t(\gamma), Q_t] = \text{corr}[U_t(\gamma), Q_t]\text{corr}[W_t(\gamma), U_t(\gamma)], \tag{A.8}$$

so that

$$\text{corr}^2[W_t(\gamma), Q_t] = \text{corr}^2[U_t(\gamma), Q_t]\text{corr}^2[W_t(\gamma), U_t(\gamma)]. \tag{A.9}$$

Note that for each γ,

$$\text{corr}^2[W_t(\gamma), U_t(\gamma)] < 1 \tag{A.10}$$

by Cauchy-Schwarz's inequality: for any c, $W_t(\gamma) \neq c \cdot U_t(\gamma)$ with probability 1. Next, we suppose that γ, $\gamma' \in \mathbf{\Gamma}$ and $\gamma' < \gamma$. If γ' is close to γ_0, we can approximate $U_t(\gamma')$ using Taylor's expansion: $U_t(\gamma') = U_t(\gamma) + (\gamma' - \gamma)W_t(\gamma) + o_p(|\gamma' - \gamma|)$. This implies that

$$\begin{aligned} g(\gamma') =& \frac{\text{cov}[U_t(\gamma), Q_t] + (\gamma' - \gamma)\text{cov}[W_t(\gamma), Q_t]}{\{\text{var}[U_t(\gamma)] + 2(\gamma' - \gamma)\text{cov}[U_t(\gamma), W_t(\gamma)] + (\gamma' - \gamma)^2\text{var}[W_t(\gamma)]\}^{1/2}\text{var}[Q_t]^{1/2}} + o(|\gamma' - \gamma|) \\ =& \frac{\text{cov}[U_t(\gamma), Q_t]}{\text{var}[U_t(\gamma)]^{1/2}\text{var}[Q_t]^{1/2}} = g(\gamma). \end{aligned}$$

This equality can also be equivalently stated as

$$\begin{aligned} &2\left\{\frac{\text{corr}[W_t(\gamma), Q_t] - \text{corr}[U_t(\gamma), Q_t]\text{corr}[W_t(\gamma), U_t(\gamma)]}{\text{corr}[U_t(\gamma), Q_t]}\right\} \\ &\qquad = (\gamma' - \gamma)\frac{\text{var}[W_t(\gamma)]^{1/2}}{\text{var}[U_t(\gamma)]^{1/2}}\left(1 - \frac{\text{corr}^2[W_t(\gamma), Q_t]}{\text{corr}^2[U_t(\gamma), Q_t]}\right) + o(|\gamma' - \gamma|). \end{aligned}$$

Note that the left side of this equality is zero by (A.8), implying that $\text{corr}^2[W_t(\gamma), Q_t] = \text{corr}^2[U_t(\gamma), Q_t]$ because $\gamma' < \gamma$. This fact, (A.9), and (A.10) imply that $\text{cov}[W_t(\cdot), Q_t] = \text{cov}[U_t(\cdot), Q_t] = 0$. Therefore, for each $\gamma \in \mathbf{\Gamma}$,

$$\frac{d^j}{d\gamma^j}\text{cov}[U_t(\gamma), Q_t] = \mathbb{E}[X_t^\gamma \log^j(X_t)Q_t] - \mathbb{E}[Q_t\mathbf{Z}_t]'\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbb{E}[\mathbf{Z}_t X_t^\gamma \log^j(X_t)] = 0.$$

This implies that for any $j \leq j_*$,

$$\lim_{\gamma \to 0} \frac{d^j}{d\gamma^j} \mathrm{cov}[U_t(\gamma), Q_t] = \mathbb{E}[\log^j(X_t) Q_t] - \mathbb{E}[Q_t \mathbf{Z}_t]' \mathbb{E}[\mathbf{Z}_t \mathbf{Z}_t']^{-1} \mathbb{E}[\mathbf{Z}_t \log^j(X_t)] = 0. \quad \text{(A.11)}$$

We here note that $\mathbb{E}[\log^{j_*}(X_t) Q_t] - \mathbb{E}[Q_t \mathbf{Z}_t]' \mathbb{E}[\mathbf{Z}_t \mathbf{Z}_t']^{-1} \mathbb{E}[\mathbf{Z}_t \log^{j_*}(X_t)] = \mathbb{E}[V_t \log^{j_*}(X_t)]$, so that (A.11) implies that $\mathbb{E}[V_t \log^{j_*}(X_t)] = 0$. This is a contradiction to (A.7). Therefore, $g(\cdot)$ must not be a constant function, and this completes the proof.

(*iii*) By the definition of the QLR test,

$$QLR_n = \frac{n(\widehat{\sigma}_{n,0}^2 - \widehat{\sigma}_{n,A}^2)}{\widehat{\sigma}_{n,0}^2} = \sup_{\gamma \in \mathbf{\Gamma}} \frac{\{(n^{-1/2}\mathbf{U} + n^{-1}\mathbf{N})'\mathbf{M}\mathbf{X}(\gamma)\}^2}{\widehat{\sigma}_{n,0}^2 (\mathbf{X}(\gamma)'\mathbf{M}\mathbf{X}(\gamma))},$$

where $\mathbf{N} := [m(X_1), \ldots, m(X_t), \ldots, m(X_n)]'$. We examine the $\widehat{\sigma}_{n,0}^2$ and $n^{-1}\mathbf{N}'\mathbf{M}\mathbf{X}(\cdot)$ as the asymptotic behaviors of the other terms are already shown when deriving the asymptotic null distribution of the QLR test.

We first examine the asymptotic behavior of $\widehat{\sigma}_{n,0}^2$. Note that

$$\widehat{\sigma}_{n,0}^2 = n^{-1}(\mathbf{U} + n^{-1/2}\mathbf{N})'\mathbf{M}(\mathbf{U} + n^{-1/2}\mathbf{N}) = n^{-1}\mathbf{U}'\mathbf{M}\mathbf{U} + 2n^{-3/2}\mathbf{N}'\mathbf{M}\mathbf{U} + n^{-2}\mathbf{N}'\mathbf{M}\mathbf{N}.$$

Here, $n^{-1}\mathbf{U}'\mathbf{U} = \sigma_*^2 + o_p(1)$, $\mathbf{N}'\mathbf{M}\mathbf{U} = O_p(n^{-1})$, and $\mathbf{N}'\mathbf{M}\mathbf{N} = O_p(n^{-1})$ by the ergodic theorem under the maintained assumptions. Therefore, $\widehat{\sigma}_{n,0}^2 = \sigma_*^2 + o_p(1)$.

Next, we examine the asymptotic behavior of $n^{-1}\mathbf{N}'\mathbf{M}\mathbf{X}(\cdot)$. Note that

$$\frac{1}{n}\mathbf{N}'\mathbf{M}\mathbf{X}(\cdot) = \frac{1}{n}\mathbf{N}'\mathbf{X}(\cdot) - \frac{1}{n}\mathbf{N}'\mathbf{Z}\left(\frac{1}{n}\mathbf{Z}'\mathbf{Z}\right)^{-1}\frac{1}{n}\mathbf{Z}'\mathbf{X}(\cdot).$$

In the proof of Theorem 1, we already showed that $n^{-1}\mathbf{Z}'\mathbf{X}(\cdot)$ uniformly converges to $\mathbb{E}[\mathbf{Z}_t X_t^{(\cdot)}]$, and $n^{-1}\mathbf{Z}'\mathbf{Z} \stackrel{\mathbb{P}}{\to} \mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']$. We also note that $n^{-1}\mathbf{N}'\mathbf{Z} = \mathbb{E}[m(X_t)\mathbf{Z}_t] + o_p(1)$ and $n^{-1}\mathbf{N}'\mathbf{X}(\cdot)$. Given the moment condition, ULLN can be applied to $\{n^{-1}\sum_{t=1}^n X_t^{(\cdot)} m(X_t)\}$. The ergodic theorem holds for $n^{-1}\sum_{t=1}^n X_t^{\gamma} m(X_t)$, and $X_t^{(\cdot)} m(X_t)$ is Lipschitz continuous because

$$|X_t^{\gamma} m(X_t) - X_t^{\gamma'} m(X_t)| \leq \sup_{\gamma \in \mathbf{\Gamma}} |X_t^{\gamma} \log(X_t)| \cdot |m(X_t)| \cdot |\gamma - \gamma'| \leq M_t^2 |\gamma - \gamma'|,$$

and $M_t^2 = O_p(1)$. These three conditions are the assumptions required in theorem 3(a) of Andrews (1992) to prove the ULLN. Thus, $\sup_{\gamma \in \mathbf{\Gamma}} |n^{-1}\mathbf{N}'\mathbf{M}\mathbf{X}(\gamma) - \mu(\gamma)| = o_p(1)$.

From these facts, it follows that $QLR_n \Rightarrow \sup_{\gamma \in \mathbf{\Gamma}} \{\mathcal{G}(\gamma) + \mu(\gamma)\}^2/\sigma^2(\gamma)$. We also note that $\mathcal{Z}(\cdot) := \mathcal{G}(\cdot)/\sigma(\cdot)$. Therefore, $QLR_n \Rightarrow \sup_{\gamma \in \mathbf{\Gamma}}\{\mathcal{Z}(\gamma) + \mu(\gamma)/\sigma(\gamma)\}^2$. This completes the proof. ■

Remarks 1. *We also note that*

$$\frac{1}{n}\mathbf{N}'\mathbf{M}\mathbf{L}_1 = \frac{1}{n}\mathbf{N}'\mathbf{L}_1 - \frac{1}{n}\mathbf{N}'\mathbf{Z}\left(\frac{1}{n}\mathbf{Z}'\mathbf{Z}\right)^{-1}\frac{1}{n}\mathbf{Z}'\mathbf{L}_1 \stackrel{\mathbb{P}}{\to} \mu_0,$$

where $\mu_0 := \mathbb{E}[m(X_t)\log(X_t)] - \mathbb{E}[m(X_t)\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbb{E}[\mathbf{Z}_t\log(X_t)]$*, and*

$$\frac{1}{n}\mathbf{N}'\mathbf{M}\mathbf{C}_1 = \frac{1}{n}\mathbf{N}'\mathbf{C}_1 - \frac{1}{n}\mathbf{N}'\mathbf{Z}\left(\frac{1}{n}\mathbf{Z}'\mathbf{Z}\right)^{-1}\frac{1}{n}\mathbf{Z}'\mathbf{C}_1 \stackrel{\mathbb{P}}{\to} \mu_1$$

where $\mu_1 := \mathbb{E}[m(X_t)X_t\log(X_t)] - \mathbb{E}[m(X_t)\mathbf{Z}_t']\mathbb{E}[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbb{E}[\mathbf{Z}_tX_t\log(X_t)]$*. Therefore,* $QLR_n^{(\gamma=0)} \stackrel{\text{A}}{\sim} (\mathcal{Z}_0 + \mu_0/\sigma_0)^2$ *and* $QLR_n^{(\gamma=1)} \stackrel{\text{A}}{\sim} (\mathcal{Z}_1 + \mu_1/\sigma_1)^2$ *under the same condition as in Theorem 5(ii).* □

Before proving the main claims in Section 3, we provide the following supplementary lemmas to assist in delivering an efficient proof.

Lemma A9. *(i)* $(n\log(n))^{-1}\sum_{t=1}^n \log(t) \to 1$*;*

(ii) $(n\log^2(n))^{-1}\sum_{t=1}^n \log^2(t) \to 1$*;*

(iii) for each $\gamma \in (-1/2, \infty)$*,* $(n^{1+2\gamma}\log(n))^{-1}\sum_{t=1}^n t^{2\gamma}\log(t) \to 1/(2\gamma+1)$*; and*

(iv) for each $\gamma \in (-1/2, \infty)$*,* $(n^{1+2\gamma}\log^2(n))^{-1}\sum_{t=1}^n t^{2\gamma}\log^2(t) \to 1/(2\gamma+1)$. □

Proof of Lemma A9: (*i* and *ii*) This immediately follows from equation (26) of Phillips (2007) by letting his $L(\cdot)$ be $\log(\cdot)$.

(*iii* and *iv*) This also immediately follows from equation (55) of Phillips (2007). ■

Lemma A10. *Given the definition of* $s_{n,t} := (t/n)$*,*

(i) for each $\gamma > -1$*,* $\frac{1}{n}\sum s_{n,t}^\gamma \to \int_0^1 s^\gamma ds = \frac{1}{1+\gamma}$*;*

(ii) for each $\gamma > -1$*,* $\frac{1}{n}\sum s_{n,t}^\gamma \log(s_{n,t}) \to \int_0^1 s^\gamma \log(s) ds = -\frac{1}{(1+\gamma)^2}$*;*

(iii) for each $\gamma > -1$*,* $\frac{1}{n}\sum s_{n,t}^\gamma \log^2(s_{n,t}) \to \int_0^1 s^\gamma \log^2(s) ds = -\frac{2}{(1+\gamma)^3}$*; and*

(iv) $\{n^{-1}\sum s_{n,t}^{(\cdot)} : \mathbf{\Gamma} \mapsto \mathbb{R}\}$ *is equicontinuous, where* $\mathbf{\Gamma}$ *is a convex and compact set in* $\mathbb{R}$. □

Proof of Lemma A10: (*i*, *ii*, and *iii*) These results are elementary.

(*iv*) We note that for some $\bar{\gamma}$ between γ and γ',

$$\left|\frac{1}{n}\sum s_{n,t}^{\gamma} - \frac{1}{n}\sum s_{n,t}^{\gamma'}\right| \leq \frac{1}{n}\sum |s_{n,t}^{\bar{\gamma}}| \cdot |\log(s_{n,t})| \cdot |\gamma - \gamma'| \leq \frac{1}{n}\sum |s_{n,t}|^{\gamma_o} \cdot |\log(s_{n,t})| \cdot |\gamma - \gamma'|,$$

where $\gamma_o := \inf \mathbf{\Gamma}$. Also, $\frac{1}{n}\sum |s_{n,t}|^{\gamma_o} \cdot |\log(s_{n,t})| \to \frac{1}{\gamma_o+2}$. Therefore, for any $\epsilon > 0$, if we let δ be $\epsilon(\gamma_o + 2)$ and $|\gamma - \gamma'| < \delta$, $\limsup_{n\to\infty} |n^{-1}\sum s_{n,t}^{\gamma} - n^{-1}\sum s_{n,t}^{\gamma'}| \leq \epsilon$. This completes the proof. ■

Lemma A11. *For a strictly stationary (SS) process* $\{\mathcal{Z}_t\}$ *and a deterministic sequence* $\{\xi_{n,t}\}$*, if we suppose that* $\mathbb{E}[|\mathcal{Z}_t|] < \infty$ *and* $\lim_{n\to\infty}\sum_{t=1}^{n}\xi_{n,t} = \xi_o \in (-\infty, \infty)$, $\sum_{t=1}^{n}\mathcal{X}_{n,t} \stackrel{\text{a.s.}}{\to} \xi_o\mathbb{E}[\mathcal{Z}_t]$*, where* $\mathcal{X}_{n,t} := \xi_{n,t}\mathcal{Z}_t$. □

Proof of Lemma A11: We can apply the corollary in Billingsley (1995, p. 211). ■

Lemma A12. *We suppose that* $\{(U_t, \mathbf{D}_t')'\}$ *is an SS process. If for each* $j = 1, 2, \ldots, k$, $\mathbb{E}[D_{t,j}^4] < \infty$ *and* $\mathbb{E}[U_t^4] < \infty$*, then for each* $\gamma \in \mathbf{\Gamma}$ *with* $\inf \mathbf{\Gamma} > -1/2$*,*

(*i*) $n^{-1}\sum \mathbf{G}_{n,t}(\gamma)\mathbf{G}_{n,t}(\gamma)' \stackrel{\text{a.s.}}{\to} \widetilde{\mathbf{A}}(\gamma)$*; and*

(*ii*) $n^{-1}\sum U_t^2\mathbf{G}_{n,t}(\gamma)\mathbf{G}_{n,t}(\gamma)' \stackrel{\text{a.s.}}{\to} \widetilde{\mathbf{B}}(\gamma)$. □

Proof of Lemma A12: (*i* and *ii*) We let $\xi_{n,t}$ of Lemma A11 be $s_{n,t}^{2\gamma}/n$, $s_{n,t}^{\gamma+1}\log(s_{n,t})/n$, $s_{n,t}^{\gamma}\log(s_{n,t})/n$, $s_{n,t}^{\gamma}/n$, $s_{n,t}^{\gamma+1}/n$, $s_{n,t}^2\log^2(s_{n,t})/n$, $s_{n,t}\log^2(s_{n,t})/n$, $s_{n,t}\log(s_{n,t})/n$, $s_{n,t}^2\log(s_{n,t})/n$, $\log^2(s_{n,t})/n$, log $(s_{n,t})/n$, $s_{n,t}\log(s_{n,t})/n$, $s_{n,t}/n$, or $s_{n,t}^2/n$. Then, Lemma A10 implies that $\sum \xi_{n,t}$ converges to $1/(2\gamma+1)$, $-1/(\gamma+2)^2$, $-1/(\gamma+1)^2$, $1/(\gamma+1)$, $1/(\gamma+2)$, $2/27$, $1/4$, $-1/4$, $-1/9$, 2, -1, $-1/4$, $1/2$, or $1/3$, respectively. We let these limits be denoted by ξ_o. Lemma A11 implies that $\sum \xi_{n,t}\mathbf{D}_t$, $\sum \xi_{n,t}U_t^2$, and $\sum \xi_{n,t}U_t^2\mathbf{D}_t$ almost surely converge to $\xi_o\mathbb{E}[\mathbf{D}_t]$, $\xi_o\mathbb{E}[U_t^2]$ and $\xi_o\mathbb{E}[U_t^2\mathbf{D}_t]$, respectively. Finally, we note that $n^{-1}\sum \mathbf{D}_t\mathbf{D}_t' \stackrel{\text{a.s.}}{\to} \mathbb{E}[\mathbf{D}_t\mathbf{D}_t']$ and $n^{-1}\sum U_t^2\mathbf{D}_t\mathbf{D}_t' \stackrel{\text{a.s.}}{\to} \mathbb{E}[U_t^2\mathbf{D}_t\mathbf{D}_t']$ by the ergodic theorem and that $\mathbb{E}[D_{t,j}^4] < \infty$ and $\mathbb{E}[U_t^4] < \infty$. These limit results are sufficient for the desired results. ■

Lemma A13. *Given the definition of* $s_{n,t} := (t/n)$*, if for each* $j = 1, 2, \ldots, k$, $\mathbb{E}[|D_{t,j}|] < \infty$ *and* $\mathbf{\Gamma}$ *is a compact and convex subset in* $\mathbb{R}$ *such that* $\inf \mathbf{\Gamma} > -1$*,*

(*i*) $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\sum s_{n,t}^{\gamma} - \frac{1}{\gamma+1}| \to 0$*; and*

(*ii*) $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\sum s_{n,t}^{\gamma}D_{t,j} - \frac{1}{\gamma+1}\mathbb{E}[D_{t,j}]| \stackrel{\text{a.s.}}{\to} 0$. □

Proof of Lemma A13: (*i*) Lemma A10(*i* and *iv*) implies the desired result.

(*ii*) For each γ, Lemma A12(*i*) implies that $n^{-1}\sum s_{n,t}^{\gamma}D_{t,j} \stackrel{\text{a.s.}}{\to} \frac{1}{\gamma+1}\mathbb{E}[D_{t,j}]$. To show the desired result, we show the stochastic equicontinuity of $\{n^{-1}\sum s_{n,t}^{(\cdot)}D_{t,j} : \mathbf{\Gamma} \mapsto \mathbb{R}\}$. We note that

$$\left|\frac{1}{n}\sum s_{n,t}^{\gamma}D_{t,j} - \frac{1}{n}\sum s_{n,t}^{\gamma'}D_{t,j}\right| \leq \frac{1}{n}\sum |s_{n,t}|^{\gamma_o}\cdot|\log(s_{n,t})|\cdot|D_{t,j}|\cdot|\gamma-\gamma'|,$$

where $\gamma_o := \inf \mathbf{\Gamma}$. This implies that for any $\epsilon > 0$,

$$\limsup_{n\to\infty} P\left(\sup_{|\gamma-\gamma'|<\delta}\left|\frac{1}{n}\sum s_{n,t}^{\gamma}D_{t,j} - \frac{1}{n}\sum s_{n,t}^{\gamma'}D_{t,j}\right| > \epsilon\right)$$
$$\leq \limsup_{n\to\infty} P\left(\frac{1}{n}\sum |s_{n,t}|^{\gamma_o}\cdot|\log(s_{n,t})|\cdot|D_{t,j}|\cdot\delta > \epsilon\right).$$

Therefore, if δ is sufficiently small, the right side can be made smaller than ϵ by using Fatou's lemma since $n^{-1}\sum|s_{n,t}|^{\gamma_o}\cdot|\log(s_{n,t})| \to 1/(\gamma_o+2)$, implying that $n^{-1}\sum|s_{n,t}|^{\gamma_o}\cdot|\log(s_{n,t})|\cdot|D_{t,j}| \overset{\text{a.s.}}{\to} [|D_{t,j}|]1/(\gamma_o+2)$ by Lemma A11. The desired result follows. ■

Proof of Lemma 3: We first note that Lemmas A9 and A10 show that $(n^{1+2\gamma})^{-1}\sum t^{2\gamma} = n^{-1}\sum s_{n,t}^{2\gamma} \to \frac{1}{2\gamma+1}$, $(n^{2+\gamma}\log(n))^{-1}\sum t^{1+\gamma}\log(t) \to \frac{1}{\gamma+2}$, $(n^{1+\gamma}\log(n))^{-1}\sum t^{\gamma}\log(t) \to \frac{1}{\gamma+1}$, $(n^{1+\gamma})^{-1}\sum t^{\gamma} \to \frac{1}{\gamma+1}$, $(n^{2+\gamma})^{-1}\sum t^{\gamma+1} \to \frac{1}{\gamma+2}$, $(n^3\log^2(n))^{-1}\sum t^2\log^2(t) \to \frac{1}{3}$, $(n^2\log^2(n))^{-1}\sum t\log^2(t) \to \frac{1}{2}$, $(n^2\log(n))^{-1}\sum t\log(t) \to \frac{1}{2}$, $(n^3\log(n))^{-1}\sum t^2\log(t) \to \frac{1}{3}$, $(n\log^2(n))^{-1}\sum\log^2(t) \to 1$, $(n\log(n))^{-1}\sum\log(t) \to 1$, $n^{-2}\sum t \to \frac{1}{2}$, and $n^{-3}\sum t^2 \to \frac{1}{3}$.

We also note that $n^{-1}\sum \mathbf{D}_t\mathbf{D}_t' \overset{\text{a.s.}}{\to} \mathbb{E}[\mathbf{D}_t\mathbf{D}_t']$ by ergodicity and $\mathbb{E}[D_{t,j}^2] < \infty$. If we further let $\xi_{n,t}$ of Lemma A11 be $t^{\gamma}/n^{1+\gamma}$, $t\log(t)/(n^2\log(n))$, $\log(t)/(n\log(n))$, $1/n$, or t/n^2, then $\sum\xi_{n,t}$ converges to $1/(\gamma+1)$, $\frac{1}{2}$, 1, 1, or, $\frac{1}{2}$, respectively. These facts and Lemma A11 imply that $\sum\xi_{n,t}\mathbf{D}_t$ almost surely converges to $\frac{1}{\gamma+1}\mathbb{E}[\mathbf{D}_t]$, $\frac{1}{2}\mathbb{E}[\mathbf{D}_t]$, $\mathbb{E}[\mathbf{D}_t]$, $\mathbb{E}[\mathbf{D}_t]$, or $\frac{1}{2}\mathbb{E}[\mathbf{D}_t]$, respectively.

Therefore, $\mathbf{F}_n^{-1}\sum_{t=1}^{n}\mathbf{H}_t(\gamma)\mathbf{H}_t(\gamma)'\mathbf{F}_n^{-1} \overset{\text{a.s.}}{\to} \mathbf{\Xi}(\gamma)$, where

$$\mathbf{\Xi}(\gamma) := \begin{bmatrix} \frac{1}{2\gamma+1} & \frac{1}{\gamma+2} & \frac{1}{\gamma+1} & \frac{1}{\gamma+1} & \frac{1}{\gamma+2} & \frac{1}{\gamma+1}\mathbb{E}[\mathbf{D}_t'] \\ \frac{1}{\gamma+2} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{2}\mathbb{E}[\mathbf{D}_t'] \\ \frac{1}{\gamma+1} & \frac{1}{2} & 1 & 1 & \frac{1}{2} & \mathbb{E}[\mathbf{D}_t'] \\ \frac{1}{\gamma+1} & \frac{1}{2} & 1 & 1 & \frac{1}{2} & \mathbb{E}[\mathbf{D}_t'] \\ \frac{1}{\gamma+2} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{2}\mathbb{E}[\mathbf{D}_t'] \\ \frac{1}{\gamma+1}\mathbb{E}[\mathbf{D}_t] & \frac{1}{2}\mathbb{E}[\mathbf{D}_t] & \mathbb{E}[\mathbf{D}_t] & \mathbb{E}[\mathbf{D}_t] & \frac{1}{2}\mathbb{E}[\mathbf{D}_t] & \mathbb{E}[\mathbf{D}_t\mathbf{D}_t'] \end{bmatrix}.$$

The limit $\mathbf{\Xi}(\gamma)$ is a singular matrix because the second column of the limit matrix is identical to the fifth column, and its third column is identical to the fourth column. This completes the proof. ■

We define the matrices relevant to Theorem 5 in the same way as in Section 2. That is, for each $\gamma \in \mathbf{\Gamma}$, $\mathbf{T}(\gamma) := [s_{n,1}^{\gamma}, \ldots, s_{n,n}^{\gamma}]'$ and $\mathbf{M} := \mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'$ with $\mathbf{Z}_{n,t}'$ as the t-th row vector of $\mathbf{Z}$.

Proof of Theorem 6: (i) We note that the QLR test statistic under $\widetilde{\mathcal{H}}_0$ is equal to

$$\sup_{\gamma \in \mathbf{\Gamma}} \frac{\{\mathbf{T}(\gamma)'\mathbf{M}\mathbf{U}\}^2}{\{\hat{\sigma}_{n,0}^2 \mathbf{T}(\gamma)'\mathbf{M}\mathbf{T}(\gamma)\}} \tag{A.12}$$

by applying Theorem 4. This result follows simply by replacing $\mathbf{X}(\gamma)$ of Theorem 4 with $\mathbf{T}(\gamma)$. In particular, if we let $\widetilde{\mathbf{L}}_1 := [\log(s_{n,1}), \ldots, \log(s_{n,1})]'$ and $\widetilde{\mathbf{C}}_1 := [s_{n,1}\log(s_{n,1}), \ldots, s_{n,n}\log(s_{n,n})]'$, the QLR test is equal to

$$\frac{\{\widetilde{\mathbf{L}}_1'\mathbf{M}\mathbf{U}\}^2}{\hat{\sigma}_{n,0}^2\{\widetilde{\mathbf{L}}_1'\mathbf{M}\widetilde{\mathbf{L}}_1\}} \quad \text{and} \quad \frac{\{\widetilde{\mathbf{C}}_1'\mathbf{M}\mathbf{U}\}^2}{\hat{\sigma}_{n,0}^2\{\widetilde{\mathbf{C}}_1'\mathbf{M}\widetilde{\mathbf{C}}_1\}} \tag{A.13}$$

under $\widetilde{\mathcal{H}}_0''$ and $\widetilde{\mathcal{H}}_0'''$, respectively. We separate the proof into three parts: (a), (b), and (c). In (a) and (b) we examine the denominators and the numerators of the statistics in (A.12) and (A.13), respectively, so that the asymptotic null behavior of the QLR test can be revealed by joint convergence. In (c) we derive the covariance structure given in the theorem.

(a) We examine the denominators of the statistics in (A.12) and (A.13). It is elementary to show that $\widehat{\sigma}_{n,0}^2 \stackrel{\text{a.s.}}{\to} \sigma_*^2$ under $\widetilde{\mathcal{H}}_0$. Next note that Lemma A12(i) implies that $n^{-1}\widetilde{\mathbf{L}}'\mathbf{M}\widetilde{\mathbf{L}} \stackrel{\text{a.s.}}{\to} 2 - \widetilde{\mathbf{A}}_{2,1}'\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{2,1}$ and $n^{-1}\widetilde{\mathbf{C}}'\mathbf{M}\widetilde{\mathbf{C}} \stackrel{\text{a.s.}}{\to} 2/27 - \widetilde{\mathbf{A}}_{3,1}'\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{3,1}$, where

$$\widetilde{\mathbf{A}}_{2,1} := \begin{bmatrix} -1 \\ -\frac{1}{4} \\ -\mathbb{E}[\mathbf{D}_t] \end{bmatrix}, \quad \widetilde{\mathbf{A}}_{3,1} := \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{9} \\ -\frac{1}{4}\mathbb{E}[\mathbf{D}_t] \end{bmatrix}, \quad \text{and} \quad \widetilde{\mathbf{A}}_{1,1} := \begin{bmatrix} 1 & \frac{1}{2} & \mathbb{E}[\mathbf{D}_t'] \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2}\mathbb{E}[\mathbf{D}_t'] \\ \mathbb{E}[\mathbf{D}_t] & \frac{1}{2}\mathbb{E}[\mathbf{D}_t] & \mathbb{E}[\mathbf{D}_t\mathbf{D}_t'] \end{bmatrix}.$$

We finally examine the denominator of $\left\{n^{-1/2}\mathbf{T}(\cdot)'\mathbf{M}\mathbf{U}\right\}/\{\hat{\sigma}_{n,0}^2 n^{-1}\mathbf{T}(\cdot)'\mathbf{M}\mathbf{T}(\cdot)\}^{1/2}$. Observer that $n^{-1}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{T}(\gamma) = n^{-1}\mathbf{T}(\gamma)'\mathbf{T}(\gamma) - n^{-1}\mathbf{T}(\gamma)'\mathbf{Z}(n^{-1}\mathbf{Z}'\mathbf{Z})^{-1}n^{-1}\mathbf{Z}'\mathbf{T}(\gamma)$, and Lemma A12($i$) implies that $n^{-1}\mathbf{T}(\gamma)'\mathbf{T}(\gamma)$, $n^{-1}\mathbf{Z}'\mathbf{T}(\gamma)$, and $n^{-1}\mathbf{Z}'\mathbf{Z}$ almost surely converges to $\widetilde{\mathbf{A}}_{4,4}(\gamma) := \frac{1}{2\gamma+1}$, $\widetilde{\mathbf{A}}_{4,1}(\gamma)$, and $\widetilde{\mathbf{A}}_{1,1}$, respectively, where $\widetilde{\mathbf{A}}_{4,1}(\gamma) := [\frac{1}{\gamma+1}, \frac{1}{\gamma+2}, \frac{1}{\gamma+1}\mathbb{E}[\mathbf{D}_t']]'$. Furthermore, Lemma A12(i and ii) implies that $\sup_{\gamma \in \mathbf{\Gamma}} |n^{-1}\mathbf{T}(\gamma)'\mathbf{T}(\gamma) - 1/(2\gamma+1)| \stackrel{\text{a.s.}}{\to} 0$ and $\sup_{\gamma \in \mathbf{\Gamma}} \|n^{-1}\mathbf{Z}'\mathbf{T}(\gamma) - \widetilde{\mathbf{A}}_{4,1}(\gamma)\|_\infty \stackrel{\text{a.s.}}{\to} 0$. Therefore,

$\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{T}(\gamma) - \{1/(2\gamma+1) - \widetilde{\mathbf{A}}_{4,1}(\gamma)'\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{4,1}(\gamma)\}| \overset{\text{a.s.}}{\to} 0$, since

$$\begin{aligned}\sup_{\gamma\in\mathbf{\Gamma}} &\left|n^{-1}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{T}(\gamma) - \{1/(2\gamma+1) - \widetilde{\mathbf{A}}_{4,1}(\gamma)'\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{4,1}(\gamma)\}\right| \\ &\leq \sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{T}(\gamma)'\mathbf{T}(\gamma) - 1/(2\gamma+1)| + \sup_{\gamma\in\mathbf{\Gamma}} \left|\{n^{-1}\mathbf{T}(\gamma)'\mathbf{Z} - \widetilde{\mathbf{A}}_{4,1}(\gamma)\}'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{T}(\gamma)\right| \\ &\quad + \sup_{\gamma\in\mathbf{\Gamma}} \left|\{\widetilde{\mathbf{A}}_{4,1}(\gamma)'^{-1}\mathbf{Z}'\mathbf{Z})^{-1} - \widetilde{\mathbf{A}}_{1,1}^{-1}\}\{n^{-1}\mathbf{Z}'\mathbf{T}(\gamma)\}\right| \\ &\quad + \sup_{\gamma\in\mathbf{\Gamma}} \left|\{\widetilde{\mathbf{A}}_{4,1}(\gamma)'\{\widetilde{\mathbf{A}}_{1,1}^{-1}\}\{n^{-1}\mathbf{T}(\gamma)'\mathbf{Z} - \widetilde{\mathbf{A}}_{4,1}(\gamma)\}\right|,\end{aligned}$$

and each element on the right side almost surely converges to zero. This shows that $n^{-1}\mathbf{T}(\cdot)'\mathbf{M}\mathbf{T}(\cdot)$ obeys the ULLN. We further note that $\widetilde{\mathbf{A}}_{4,4}(\gamma)-\widetilde{\mathbf{A}}_{4,1}(\gamma)'\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{4,1}(\gamma) = \sigma_*^2\gamma^2(\gamma-1)^2/\{(\gamma+1)^2(\gamma+2)^2(2\gamma+1)\}$ by using the definition of $\widetilde{\mathbf{A}}_{4,4}(\gamma)$, $\widetilde{\mathbf{A}}_{4,1}(\gamma)$, and $\widetilde{\mathbf{A}}_{1,1}$. For notational simplicity, let the right side be $\sigma^2(\gamma)$. If we combine all these limit results, it follows that

$$\begin{aligned}&\left\{\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\widehat{\sigma}_{n,0}^2\mathbf{T}(\gamma)'\mathbf{M}\mathbf{T}(\gamma) - \sigma^2(\gamma)|, n^{-1}\widehat{\sigma}_{n,0}^2\widetilde{\mathbf{L}}'\mathbf{M}\widetilde{\mathbf{L}}, n^{-1}\widehat{\sigma}_{n,0}^2\widetilde{\mathbf{C}}'\mathbf{M}\widetilde{\mathbf{C}},\right\} \\ &\qquad \overset{\text{a.s.}}{\to} \left\{0, \sigma_*^2(2-\widetilde{\mathbf{A}}_{2,1}'\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{2,1}), \sigma_*^2(2/27-\widetilde{\mathbf{A}}_{3,1}'\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{3,1})\right\}. \qquad (A.14)\end{aligned}$$

(*b*) We next examine the numerators of the statistics in (A.12) and (A.13). We first show that for each γ, $\{n^{-1/2}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{U}, n^{-1/2}\widetilde{\mathbf{L}}_1'\mathbf{M}\mathbf{U}, n^{-1/2}\widetilde{\mathbf{C}}_1'\mathbf{M}\mathbf{U}\}$ weakly converges to a multivariate normal variate. We note that $n^{-1/2}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{U} = n^{-1/2}\mathbf{T}(\gamma)'\mathbf{U} - (n^{-1}\mathbf{T}(\gamma)'\mathbf{Z})(n^{-1}\mathbf{Z}'\mathbf{Z})^{-1}(n^{-1/2}\mathbf{Z}'\mathbf{U})$, $n^{-1/2}\widetilde{\mathbf{C}}_1'\mathbf{M}\mathbf{U} = n^{-1/2}\widetilde{\mathbf{C}}_1'\mathbf{U} - (n^{-1}\widetilde{\mathbf{C}}_1'\mathbf{Z})(n^{-1}\mathbf{Z}'\mathbf{Z})^{-1}(n^{-1/2}\mathbf{Z}'\mathbf{U})$, and $n^{-1/2}\widetilde{\mathbf{L}}_1'\mathbf{M}\mathbf{U} = n^{-1/2}\widetilde{\mathbf{L}}_1'\mathbf{U} - (n^{-1}\widetilde{\mathbf{L}}_1'\mathbf{Z})(n^{-1}\mathbf{Z}'\mathbf{Z})^{-1}$ $(n^{-1/2}\mathbf{Z}'\mathbf{U})$, and (A.14) implies that for each γ, $\{n^{-1}\mathbf{T}(\gamma)'\mathbf{Z}, n^{-1}\widetilde{\mathbf{L}}_1'\mathbf{Z}, n^{-1}\widetilde{\mathbf{C}}_1'\mathbf{Z}, n^{-1}\mathbf{Z}'\mathbf{Z}\}$ has its own almost sure limit. Furthermore, for each $\gamma \in \mathbf{\Gamma} \setminus \{0,1\}$, $\{U_t\mathbf{G}_{n,t}(\gamma), \mathcal{F}_t\}$ is an MDS and we can apply McLeish's (1974) CLT. Assumption 7 implies that $n^{-1}\sum \mathbb{E}[U_t^2\mathbf{G}_t(\gamma)\,\mathbf{G}_t(\gamma)']$ is uniformly positive definite with respect to n. Thus, for each γ, $n^{-1/2}\sum U_t\mathbf{G}_t(\gamma) \overset{\text{A}}{\sim} N(\mathbf{0}, \widetilde{\mathbf{B}}(\gamma))$. We also note that for each $\gamma \in \mathbf{\Gamma}$, $\sum U_t\mathbf{G}_t(\gamma) = [\mathbf{T}(\gamma)'\mathbf{U}, \widetilde{\mathbf{C}}_1'\mathbf{U}, \widetilde{\mathbf{L}}_1'\mathbf{U}, (\mathbf{Z}'\mathbf{U})']'$, so that $\{n^{-1/2}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{U}, n^{-1/2}\widetilde{\mathbf{L}}_1'\mathbf{M}\mathbf{U}, n^{-1/2}\widetilde{\mathbf{C}}_1'\mathbf{M}\mathbf{U}\}$ weakly converges to a multivariate normal vector by joint convergence. We denote this weak limit by $\{\widetilde{\mathcal{G}}(\gamma), \widetilde{\mathcal{G}}_0, \widetilde{\mathcal{G}}_1\}$.

Similarly, we have finite dimensional convergence of the vectors $\{n^{-1/2}\mathbf{T}(\cdot)'\mathbf{M}\mathbf{U}\}$. So we concentrate on showing that $\{n^{-1/2}\mathbf{T}(\cdot)'\mathbf{M}\mathbf{U}\}$ is tight. As we have already shown in (*a*) that $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{Z}'\mathbf{T}(\gamma) - \widetilde{\mathbf{A}}_{4,1}(\gamma)| \overset{\text{a.s.}}{\to} 0$ and $n^{-1}\mathbf{Z}'\mathbf{Z} \overset{\text{a.s.}}{\to} \widetilde{\mathbf{A}}_{1,1}$, if $\{n^{-1/2}\mathbf{T}(\cdot)'\mathbf{U}\}$ is tight, then $\{n^{-1/2}\mathbf{T}(\cdot)'\mathbf{M}\mathbf{U}\}$ weakly converges to a Gaussian process. Without loss of generality, we let $\gamma' > \gamma$. Then, for some $\bar{\gamma}$ between γ and γ',

$s_{n,t}^{\gamma} - s_{n,t}^{\gamma'} = s_{n,t}^{\bar{\gamma}} \log(s_{n,t}) \cdot (\gamma - \gamma') \leq s_{n,t}^{\gamma_o} |\log(s_{n,t})| \cdot |\gamma - \gamma'|$, where $\gamma_o := \inf_\gamma \mathbf{\Gamma}$, so that for any $\epsilon > 0$,

$$\begin{aligned}\limsup_{n\to\infty} P & \left(\sup_{|\gamma-\gamma'|<\delta} \left| \frac{1}{\sqrt{n}} \sum s_{n,t}^{\gamma} U_t - \frac{1}{\sqrt{n}} \sum s_{n,t}^{\gamma'} U_t \right| > \epsilon \right) \\ & \leq \limsup_{n\to\infty} P \left(\left| \frac{1}{\sqrt{n}} \sum |s_{n,t}|^{\gamma_o} \cdot |\log(s_{n,t})| \cdot U_t \right| \cdot \delta > \epsilon \right).\end{aligned}$$

We further note that $n^{-1/2} \sum |s_{n,t}|^{\gamma_o} |\log(s_{n,t})| U_t \overset{\text{A}}{\sim} N(0, 2\sigma_*^2/(1+2\gamma_o)^3)$. Thus, if δ is sufficiently small, the right side can be made as small as desired. Hence, the random process sequence $\{n^{-1/2}\mathbf{T}(\cdot)'\mathbf{U}\}$ is tight, so that

$$\{n^{-1/2}\mathbf{T}(\cdot)'\mathbf{MU}, n^{-1/2}\widetilde{\mathbf{L}}'\mathbf{MU}, n^{-1/2}\widetilde{\mathbf{C}}'\mathbf{MU}\} \Rightarrow \{\widetilde{\mathcal{G}}(\cdot), \widetilde{\mathcal{G}}_0, \widetilde{\mathcal{G}}_1\}. \tag{A.15}$$

(*c*) Finally, we derive the covariance structure of the power Gaussian process. We first examine the limit covariance structure of the numerator in $\{\mathbf{T}(\cdot)'\mathbf{MU}\} / \{\hat{\sigma}_{n,0}^2 \mathbf{T}(\cdot)'\mathbf{MT}(\cdot)\}^{1/2}$. Note that $\mathbf{T}(\gamma)'\mathbf{MU} = \mathbf{T}(\gamma)'\mathbf{U} - (\mathbf{T}(\gamma)'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{U})$, so that

$$\begin{aligned}\mathbf{T}(\gamma)'\mathbf{MUU}'\mathbf{MT}(\gamma') =& (\mathbf{T}(\gamma)'\mathbf{U})(\mathbf{U}'\mathbf{T}(\gamma')) - (\mathbf{T}(\gamma)'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\{(\mathbf{Z}'\mathbf{U})(\mathbf{U}'\mathbf{T}(\gamma')\} \\ & - \left\{(\mathbf{T}(\gamma)'\mathbf{U})\left(\mathbf{U}'\mathbf{Z}\right)\right\}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{T}(\gamma') \\ & + (\mathbf{T}(\gamma)'\mathbf{Z})(\mathbf{Z}'\mathbf{Z})^{-1}\{(\mathbf{Z}'\mathbf{U})(\mathbf{U}'\mathbf{Z})\}(\mathbf{Z}'\mathbf{Z})^{-1}(\mathbf{Z}'\mathbf{T}(\gamma')).\end{aligned}$$

Lemma A12 shows that $n^{-1}\mathbf{T}(\gamma)'\mathbf{Z} \overset{\text{a.s.}}{\to} \widetilde{\mathbf{A}}_{4,1}(\gamma)'$ and $n^{-1}\mathbf{Z}'\mathbf{Z} \overset{\text{a.s.}}{\to} \widetilde{\mathbf{A}}_{1,1}$, respectively. This implies that

$$\begin{aligned}n^{-1}\mathbf{T}(\gamma)'\mathbf{MUU}'\mathbf{MT}(\gamma') =& n^{-1}(\mathbf{T}(\gamma)'\mathbf{U})(\mathbf{U}'\mathbf{T}(\gamma')) - n^{-1}\widetilde{\mathbf{A}}_{4,1}(\gamma)'\widetilde{\mathbf{A}}_{1,1}^{-1}\{(\mathbf{Z}'\mathbf{U})(\mathbf{U}'\mathbf{T}(\gamma')\} \\ & - n^{-1}\left\{(\mathbf{T}(\gamma)'\mathbf{U})\left(\mathbf{U}'\mathbf{Z}\right)\right\}\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{4,1}(\gamma') \\ & + n^{-1}\widetilde{\mathbf{A}}_{4,1}(\gamma)'\widetilde{\mathbf{A}}_{1,1}^{-1}\{(\mathbf{Z}'\mathbf{U})(\mathbf{U}'\mathbf{Z})\}\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{4,1}(\gamma') + o_p(1).\end{aligned} \tag{A.16}$$

To find the covariance structure of the limit process of $n^{-1/2}\mathbf{T}(\cdot)'\mathbf{MU}$, we consider the limit expectations of the terms on the right side of (A.16). First,

$$n^{-1}\mathbb{E}\left[\mathbf{T}(\gamma)'\mathbf{UU}'\mathbf{T}(\gamma')\right] = n^{-1}\sum s_{n,t}^{\gamma+\gamma'}\mathbb{E}[U_t^2] \to \frac{\sigma_*^2}{\gamma+\gamma'+1}, \tag{A.17}$$

using Lemma A10(*i*) and the fact that $\{U_t, \mathcal{F}_t\}$ is an MDS. Second,

$$n^{-1}\mathbb{E}\left[(\mathbf{Z}'\mathbf{U})(\mathbf{U}'\mathbf{T}(\gamma'))\right] = n^{-1}\sum s_{n,t}^{\gamma'}\mathbb{E}[U_t^2\mathbf{Z}_{n,t}] \to \widetilde{\mathbf{B}}_{4,1}(\gamma') := \left[\frac{\sigma_*^2}{\gamma'+1}, \frac{\sigma_*^2}{\gamma'+2}, \frac{1}{\gamma'+1}\mathbb{E}[U_t^2\mathbf{D}_t']\right]',$$

and so

$$\widetilde{\mathbf{A}}_{4,1}(\gamma)'\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{B}}_{4,1}(\gamma') = \frac{\sigma_*^2(4\gamma\gamma' + 2\gamma + 2\gamma' + 4)}{(\gamma+1)(\gamma+2)(\gamma'+1)(\gamma'+2)}, \tag{A.18}$$

which is symmetric between γ and γ', thereby giving the limit of the expectation of the second and third terms of (A.16). Next observe that

$$n^{-1}\mathbb{E}[(\mathbf{Z}'\mathbf{U})(\mathbf{U}'\mathbf{Z})] = n^{-1}\sum \mathbb{E}[U_t^2\mathbf{Z}_{n,t}\mathbf{Z}_{n,t}'] \to \widetilde{\mathbf{B}}_{1,1} := \begin{bmatrix} \sigma_*^2 & \frac{1}{2}\sigma_*^2 & \mathbb{E}[U_t^2\mathbf{D}_t'] \\ \frac{1}{2}\sigma_*^2 & \frac{1}{3}\sigma_*^2 & \frac{1}{2}\mathbb{E}[U_t^2\mathbf{D}_t'] \\ \mathbb{E}[U_t^2\mathbf{D}_t] & \frac{1}{2}\mathbb{E}[U_t^2\mathbf{D}_t] & \mathbb{E}[U_t^2\mathbf{D}_t\mathbf{D}_t'] \end{bmatrix}$$

using Lemma A10(*ii*) and the fact that $\{U_t, \mathcal{F}_t\}$ is an MDS. Then,

$$\begin{aligned} n^{-1}&\widetilde{\mathbf{A}}_{4,1}(\gamma)'\widetilde{\mathbf{A}}_{1,1}^{-1}\mathbb{E}\{(\mathbf{Z}'\mathbf{U})(\mathbf{U}'\mathbf{Z})\}\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{4,1}(\gamma') \\ &= \widetilde{\mathbf{A}}_{4,1}(\gamma)'\widetilde{\mathbf{A}}_{1,1}^{-1}\left\{\mathbb{E}[n^{-1}\sum U_t^2\mathbf{Z}_{n,t}\mathbf{Z}_{n,t}']\right\}\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{4,1}(\gamma') \\ &\to \widetilde{\mathbf{A}}_{4,1}(\gamma)'\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{B}}_{1,1}\widetilde{\mathbf{A}}_{1,1}^{-1}\widetilde{\mathbf{A}}_{4,1}(\gamma') = \frac{\sigma_*^2(4\gamma\gamma' + 2\gamma + 2\gamma' + 4)}{(\gamma+1)(\gamma+2)(\gamma'+1)(\gamma'+2)}. \end{aligned} \tag{A.19}$$

We combine all the limit results in (A.17), (A.18), and (A.19) to obtain the following limiting covariance kernel of the process $n^{-1/2}\mathbf{T}(\cdot)'\mathbf{M}\mathbf{U}$

$$\sigma(\gamma, \gamma') := \frac{\sigma_*^2\gamma\gamma'(\gamma-1)(\gamma'-1)}{(\gamma+1)(\gamma+2)(\gamma'+1)(\gamma'+2)(\gamma+\gamma'+1)}.$$

The limit behavior of the denominator of $\{\mathbf{T}(\cdot)'\mathbf{M}\mathbf{U}\}/\{\hat{\sigma}_{n,0}^2\mathbf{T}(\cdot)'\mathbf{M}\mathbf{T}(\cdot)\}^{1/2}$ is already given in (*a*). That is, $\hat{\sigma}_{n,0}^2 n^{-1}\mathbf{T}(\cdot)'\mathbf{M}\mathbf{T}(\cdot)$ almost surely converges to $\sigma^2(\cdot)$ uniformly on $\mathbf{\Gamma}$. Therefore, using the definition $c(\gamma, \gamma')$, the covariance kernel of the limit $\widetilde{\mathcal{Z}}(\gamma)$ of the process $\{\mathbf{T}(\cdot)'\mathbf{M}\mathbf{U}\}/\{\hat{\sigma}_{n,0}^2\mathbf{T}(\cdot)'\mathbf{M}\mathbf{T}(\cdot)\}^{1/2}$ is given by

$$\tilde{\kappa}\left(\gamma, \gamma'\right) = \mathbb{E}[\widetilde{\mathcal{Z}}(\gamma)\widetilde{\mathcal{Z}}(\gamma')] = \frac{\sigma(\gamma, \gamma')}{\sqrt{\sigma^2(\gamma)}\sqrt{\sigma^2(\gamma')}} = c(\gamma, \gamma')\frac{(2\gamma+1)^{1/2}(2\gamma'+1)^{1/2}}{(\gamma+\gamma'+1)},$$

as stated.

(*ii*) We note that

$$\widehat{\sigma}_{n,0}^2 - \widehat{\sigma}_{n,A}^2 = \sup_{\gamma \in \mathbf{\Gamma}} \frac{\{n^{-1}(\mathbf{U} + \mathbf{G}(\gamma_*))'\mathbf{M}\mathbf{T}(\gamma)\}^2}{(n^{-1}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{T}(\gamma))}$$

where $\mathbf{G}(\gamma_*) := \beta_*[1, 2^{\gamma_*}, \ldots, t^{\gamma_*}, \ldots, (n-1)^{\gamma_*}, n^{\gamma_*}]'$. By (A.15), $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{U}'\mathbf{M}\mathbf{T}(\gamma)| = o_p(1)$, and (A.14) implies that $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{T}(\gamma) - \sigma^2(\gamma)/\sigma_*^2| = o_p(1)$. We also note that for each γ,

$\sigma^2(\gamma)/\sigma_*^2 = g(\gamma,\gamma)$. Thus, $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{T}(\gamma) - g(\gamma,\gamma)| = o_p(1)$. Furthermore, $n^{-1}\mathbf{G}(\gamma_*)'\mathbf{M}\mathbf{T}(\gamma) = \beta_* n^{\gamma_*} n^{-1}\mathbf{T}(\gamma_*)\mathbf{M}\mathbf{T}(\gamma)$, so that it now follows that $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1-\gamma_*}\mathbf{G}(\gamma_*)'\mathbf{M}\mathbf{T}(\gamma) - \beta_* g(\gamma_*,\gamma)| = o_p(1)$. Therefore,

$$\widehat{\sigma}_{n,o}^2 - \widehat{\sigma}_{n,A}^2 = \sup_{\gamma\in\mathbf{\Gamma}} \beta_*^2 n^{2\gamma_*} \frac{g^2(\gamma_*,\gamma)}{g(\gamma,\gamma)}[1+o_p(1)]. \tag{A.20}$$

Next, we note that $\widehat{\sigma}_{n,0}^2 = n^{-1}(\mathbf{U}+\mathbf{G}(\gamma_*))'\mathbf{M}(\mathbf{U}+\mathbf{G}(\gamma_*))$, so that it now follows that

$$\widehat{\sigma}_{n,0}^2 = \sigma_*^2 + \beta_*^2 n^{2\gamma_*} g(\gamma_*,\gamma_*)[1+o_p(1)], \tag{A.21}$$

from the fact that $n^{-1}\mathbf{U}'\mathbf{M}\mathbf{U} = \sigma_*^2 + o_p(1)$, $n^{-1}\mathbf{G}(\gamma_*)'\mathbf{M}\mathbf{U} = \beta_* n^{\gamma_*} n^{-1}\mathbf{T}(\gamma_*)\mathbf{M}\mathbf{U} = O_p(n^{\gamma_*-1/2})$, and $n^{-1}\mathbf{G}(\gamma_*)'\mathbf{M}\,\mathbf{G}(\gamma_*) = \beta_*^2 n^{2\gamma_*} n^{-1}\mathbf{T}(\gamma_*)\mathbf{M}\mathbf{T}(\gamma_*) = \beta_*^2 n^{2\gamma_*}[g(\gamma_*,\gamma_*)+o_p(1)]$.

(*ii.a*) We now combine (A.20) and (A.21) and obtain that

$$\frac{QLR_n}{n} = \frac{\widehat{\sigma}_{n,0}^2 - \widehat{\sigma}_{n,A}^2}{\widehat{\sigma}_{n,0}^2} = \sup_{\gamma\in\mathbf{\Gamma}} \left(\frac{\beta_*^2 n^{2\gamma_*} \frac{g^2(\gamma_*,\gamma)}{g(\gamma,\gamma)}[1+o_p(1)]}{\sigma_*^2 + \beta_*^2 n^{2\gamma_*} g(\gamma_*,\gamma_*)[1+o_p(1)]} \right) = \sup_{\gamma\in\mathbf{\Gamma}} \frac{g^2(\gamma_*,\gamma)}{g(\gamma,\gamma)g(\gamma_*,\gamma_*)}[1+o_p(1)]$$

from the fact that $\gamma_* > 0$. Finally, the given functional form of $g(\gamma_*,\gamma)$ yields that

$$\sup_{\gamma\in\mathbf{\Gamma}} \frac{g^2(\gamma_*,\gamma)}{g(\gamma,\gamma)g(\gamma_*,\gamma_*)}[1+o_p(1)] = \frac{g^2(\gamma_*,\bar{\gamma})}{g(\bar{\gamma},\bar{\gamma})g(\gamma_*,\gamma_*)}[1+o_p(1)],$$

as desired.

(*ii.b*) We now combine (A.20) and (A.21) and obtain that

$$\frac{QLR_n}{n^{1+2\gamma_*}} = \frac{\widehat{\sigma}_{n,0}^2 - \widehat{\sigma}_{n,A}^2}{n^{2\gamma_*}\widehat{\sigma}_{n,0}^2} = \sup_{\gamma\in\mathbf{\Gamma}} \left(\frac{\beta_*^2 \frac{g^2(\gamma_*,\gamma)}{g(\gamma,\gamma)}[1+o_p(1)]}{\sigma_*^2 + \beta_*^2 n^{2\gamma_*} g(\gamma_*,\gamma_*)[1+o_p(1)]} \right) = \sup_{\gamma\in\mathbf{\Gamma}} \frac{\beta_*^2 g^2(\gamma_*,\gamma)}{\sigma_*^2 g(\gamma,\gamma)}[1+o_p(1)]$$

from the fact that $\gamma_* \in (-\frac{1}{2}, 0)$. Finally, the given functional form of $g(\gamma_*,\gamma)$ yields that

$$\sup_{\gamma\in\mathbf{\Gamma}} \frac{\beta_*^2 g^2(\gamma_*,\gamma)}{\sigma_*^2 g(\gamma,\gamma)}[1+o_p(1)] = \frac{\beta_*^2 g^2(\gamma_*,\bar{\gamma})}{\sigma_*^2 g(\bar{\gamma},\bar{\gamma})}[1+o_p(1)],$$

as desired.

(*iii*) From the definition of the QLR test, if we let $\mathbf{Q} := [m(1), m(2), \ldots, m(n-1), m(n)]'$,

$$\frac{QLR_n}{n} = \frac{\widehat{\sigma}_{n,0}^2 - \widehat{\sigma}_{n,A}^2}{\widehat{\sigma}_{n,0}^2} = \sup_{\gamma\in\mathbf{\Gamma}} \frac{\{n^{-1}(\mathbf{U}+\mathbf{Q})'\mathbf{M}\mathbf{T}(\gamma)\}\{n^{-1}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{T}(\gamma)\}^{-1}\{n^{-1}\mathbf{T}(\gamma)'\mathbf{M}(\mathbf{U}+\mathbf{Q})\}}{n^{-1}(\mathbf{U}+\mathbf{Q})'\mathbf{M}(\mathbf{U}+\mathbf{Q})}.$$

In the proof of (*ii*), we already saw that $n^{-1}\mathbf{U}'\mathbf{M}\mathbf{U} = \sigma_*^2 + o_p(1)$, $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{T}(\gamma) - g(\gamma,\gamma)| = o_p(1)$, and $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{U}'\mathbf{M}\mathbf{T}(\gamma)| = o_p(1)$. We, therefore, examine the asymptotic behaviors of the other terms that constitute the QLR test.

First, we examine the asymptotic behavior of $\mathbf{Q}'\mathbf{M}\mathbf{T}(\cdot)$. Note that

$$\frac{\mathbf{Q}'\mathbf{M}\mathbf{T}(\gamma)}{nm(n)\varepsilon(n)} = \frac{1}{n\varepsilon(n)}\mathbf{L}'\mathbf{T}(\gamma) - \frac{1}{n\varepsilon(n)}\mathbf{L}'\mathbf{Z}(n^{-1}\mathbf{Z}\mathbf{Z}')^{-1}n^{-1}\mathbf{Z}'\mathbf{T}(\gamma),$$

where $\varepsilon(x) := x\cdot m'(x)/m(x)$, $\mathbf{L} := [\ell_n(1)-1,\ldots,\ell_n(t)-1,\ldots,\ell_n(n)-1]'$, and $\ell_n(t) := m(t)/m(n)$. In the proof of (*i*), we already showed that $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{Z}'\mathbf{T}(\gamma) - \widetilde{\mathbf{A}}_{4,1}(\gamma)| \overset{\text{a.s.}}{\to} 0$ and $n^{-1}\mathbf{Z}'\mathbf{Z} \overset{\text{a.s.}}{\to} \widetilde{\mathbf{A}}_{1,1}$. For the other terms, we apply lemma 4.1 of Phillips (2007) and obtain that

$$\sup_{\gamma\in\mathbf{\Gamma}} \left| \frac{1}{n\varepsilon(n)} \sum s_{n,t}^{\gamma}(\ell_n(t)-1) - \frac{1}{n}\sum s_{n,t}^{\gamma}\log(s_{n,t})\right| = o(1),$$

and that

$$\sup_{\gamma\in\mathbf{\Gamma}} \left| \frac{1}{n\varepsilon(n)} \sum s_{n,t}^{\gamma}\log(s_{n,t}) + \frac{1}{(\gamma+1)^2}\right| = o(1).$$

Therefore, $\sup_{\gamma\in\mathbf{\Gamma}} \left|(n\varepsilon(n))^{-1}\mathbf{L}'\mathbf{T}(\gamma) + 1/(\gamma+1)^2\right| = o_p(1)$. We also note that lemma 4.1 of Phillips (2007) implies that

$$\frac{1}{n\varepsilon(n)}\sum s_{n,t}(\ell_n(t)-1) = \frac{1}{n}\sum s_{n,t}\log(s_{n,t}) + o_p(1) = -\frac{1}{4} + o(1),$$

$$\frac{1}{n\varepsilon(n)}\sum \widetilde{\mathbf{D}}_t(\ell_n(t)-1) = \frac{1}{n}\sum \mathbb{E}[\widetilde{\mathbf{D}}_t]\log(s_{n,t}) + o_p(1) = -\mathbb{E}[\widetilde{\mathbf{D}}_t] + o_p(1).$$

Therefore, $(n\varepsilon(n))^{-1}\mathbf{L}'\mathbf{Z} + [1/4, \mathbb{E}[\widetilde{\mathbf{D}}_t]']' = o_p(1)$, and it follows that

$$\sup_{\gamma\in\mathbf{\Gamma}} \left| \frac{1}{nm(n)\varepsilon(n)}\mathbf{Q}'\mathbf{M}\mathbf{T}(\gamma) - p(\gamma)\right| = o_p(1) \tag{A.22}$$

by noting that $p(\gamma) := -(1+\gamma)^{-2} + \widetilde{\mathbf{A}}_{4,1}(\gamma)'(\widetilde{\mathbf{A}}_{1,1})^{-1}\widetilde{\mathbf{A}}_{5,1}$, where $\widetilde{\mathbf{A}}_{5,1} := [1/4, \mathbb{E}[\widetilde{\mathbf{D}}_t]']'$, and this implies that

$$\widehat{\sigma}_{n,0}^2 - \widehat{\sigma}_{n,A}^2 = \sup_{\gamma\in\mathbf{\Gamma}} \frac{m^2(n)\varepsilon^2(n)p(\gamma)^2}{g(\gamma,\gamma)}[1 + o_p(1)]. \tag{A.23}$$

Next, we examine the asymptotic behavior of $\mathbf{Q}'\mathbf{M}\mathbf{Q}$. Note that

$$\frac{\mathbf{Q}'\mathbf{M}\mathbf{Q}}{nm^2(n)\varepsilon^2(n)} = \frac{1}{n\varepsilon^2(n)}\mathbf{L}'\mathbf{L} - \frac{1}{n\varepsilon(n)}\mathbf{L}'\mathbf{Z}\left(\frac{1}{n}\mathbf{Z}\mathbf{Z}'\right)^{-1}\frac{1}{n\varepsilon(n)}\mathbf{Z}'\mathbf{L},$$

and we already showed that $(n\varepsilon(n))^{-1}\mathbf{L}'\mathbf{Z} + \widetilde{\mathbf{A}}_{5,1} = o_p(1)$, $\frac{1}{n}\mathbf{Z}\mathbf{Z}' \overset{\text{a.s.}}{\to} \widetilde{\mathbf{A}}_{1,1}$. Given this, lemma 7.1 of Phillips (2007) implies that $(n\varepsilon^2(n))^{-1}\mathbf{L}'\mathbf{L} = (n\varepsilon^2(n))^{-1}\sum_{t=1}^{n}(\ell_n(t)-1)^2 + o(1) = 2 + o(1)$. Therefore,

$$\frac{1}{nm^2(n)\varepsilon^2(n)}\mathbf{Q}'\mathbf{M}\mathbf{Q} = 2 - \widetilde{\mathbf{A}}_{5,1}'(\widetilde{\mathbf{A}}_{1,1})^{-1}\widetilde{\mathbf{A}}_{5,1} + o_p(1) = q + o_p(1) \tag{A.24}$$

by noting that $q := 29/16 - 25K/64$.

Third, we examine the asymptotic behavior of $\mathbf{Q}'\mathbf{M}\mathbf{U}$. Note that

$$\frac{\mathbf{Q}'\mathbf{M}\mathbf{U}}{\sqrt{n}m(n)\varepsilon(n)} = \frac{1}{\sqrt{n}\varepsilon(n)}\mathbf{L}'\mathbf{U} - \frac{1}{\sqrt{n}\varepsilon(n)}\mathbf{L}'\mathbf{Z}\left(\frac{1}{n}\mathbf{Z}\mathbf{Z}'\right)^{-1}\frac{1}{n}\mathbf{Z}'\mathbf{U}.$$

Here, $\mathbf{L}'\mathbf{Z} = O_p(n\varepsilon(n))$, $\mathbf{Z}\mathbf{Z}' = O_p(n)$, and $\mathbf{Z}'\mathbf{U} = O_p(\sqrt{n})$. Furthermore, $\mathbf{L}'\mathbf{U} = O_p(\sqrt{n}\varepsilon(n))$ by lemma 2.1 of Phillips (2007). Therefore,

$$\frac{\mathbf{Q}'\mathbf{M}\mathbf{U}}{\sqrt{n}m(n)\varepsilon(n)} = O_p(1), \tag{A.25}$$

and it follows from (A.24) and (A.25) that

$$\widehat{\sigma}_{n,0}^2 = \sigma_*^2 + m^2(n)\varepsilon^2(n)(q + o_p(1)) + o_p(1). \tag{A.26}$$

If we combine (A.23) and (A.26),

$$\frac{QLR_n}{n} = \frac{\widehat{\sigma}_{n,0}^2 - \widehat{\sigma}_{n,A}^2}{\widehat{\sigma}_{n,0}^2} = \sup_{\gamma\in\mathbf{\Gamma}} \frac{m^2(n)\varepsilon^2(n)p(\gamma)^2[1+o_p(1)]}{g(\gamma,\gamma)\{\sigma_*^2 + m^2(n)\varepsilon^2(n)[q+o_p(1)]\}}. \tag{A.27}$$

(*iii.a*) We now suppose that $n \cdot m'(n) = \varepsilon(n)m(n) \to c$. Then, (A.27) implies that

$$\frac{QLR_n}{n} = \sup_{\gamma\in\mathbf{\Gamma}} \frac{m^2(n)\varepsilon^2(n)p(\gamma)^2[1+o_p(1)]}{g(\gamma,\gamma)\{\sigma_*^2 + m^2(n)\varepsilon^2(n)[q+o_p(1)]\}} = \sup_{\gamma\in\mathbf{\Gamma}} \frac{c^2p(\gamma)^2}{g(\gamma,\gamma)\{\sigma_*^2 + c^2q\}} + o_p(1),$$

as desired.

(*iii.b*) We now suppose that $n \cdot m'(n) = \varepsilon(n)m(n) \to \infty$. Then, (A.27) implies that

$$\frac{QLR_n}{n} = \sup_{\gamma\in\mathbf{\Gamma}} \frac{m^2(n)\varepsilon^2(n)p(\gamma)^2[1+o_p(1)]}{g(\gamma,\gamma)\{\sigma_*^2 + m^2(n)\varepsilon^2(n)[q+o_p(1)]\}} = \sup_{\gamma\in\mathbf{\Gamma}} \frac{p(\gamma)^2}{q \cdot g(\gamma,\gamma)} + o_p(1),$$

as desired.

(*iv*) By the definition of the QLR test,

$$QLR_n = \sup_{\gamma \in \mathbf{\Gamma}} \frac{\{(n^{-1/2}\mathbf{U} + n^{-\gamma_*-1}\mathbf{G}(\gamma_*))'\mathbf{MT}(\gamma)\}^2}{\widehat{\sigma}_{n,0}^2\,(n^{-1}\mathbf{T}(\gamma)'\mathbf{MT}(\gamma))}.$$

We already saw that $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-1}\mathbf{T}(\gamma)'\mathbf{MT}(\gamma) - g(\gamma,\gamma)| = o_p(1)$, and (A.15) shows the weak limit of $n^{-1/2}\mathbf{U}'\mathbf{MT}(\cdot)$. We therefore here focus on the asymptotic behaviors of $\widehat{\sigma}_{n,0}^2$ and $n^{-\gamma_*-1}\mathbf{G}(\gamma_*)'\mathbf{MT}(\cdot)$.

First, we examine the asymptotic behavior of $\widehat{\sigma}_{n,0}^2$. Note that

$$\begin{aligned}\widehat{\sigma}_{n,0}^2 &= n^{-1}(\mathbf{U} + n^{-\gamma_*-1/2}\mathbf{G}(\gamma_*))'\mathbf{M}(\mathbf{U} + n^{-\gamma_*-1/2}\mathbf{G}(\gamma_*))\\ &= n^{-1}\mathbf{U}'\mathbf{MU} + 2\beta_* n^{-3/2}\mathbf{T}(\gamma_*)'\mathbf{MU} + \beta_*^2 n^{-2}\mathbf{T}(\gamma_*)'\mathbf{MT}(\gamma_*) = \sigma_*^2 + o_p(1).\end{aligned}$$

Here, $n^{-3/2}\mathbf{T}(\gamma_*)'\mathbf{MU} = O_p(n^{-1})$ and $n^{-2}\mathbf{T}(\gamma_*)'\mathbf{MT}(\gamma_*) = O_p(n^{-1})$ from (A.15) and (A.14), respectively.

Second, we examine $n^{-\gamma_*-1}\mathbf{G}(\gamma_*)'\mathbf{MT}(\cdot)$. Note that $n^{-\gamma_*-1}\mathbf{G}(\gamma_*)'\mathbf{MT}(\cdot) = \beta_* n^{-1}\mathbf{T}(\gamma_*)'\mathbf{MT}(\cdot)$, so that (A.14) implies that $\sup_{\gamma\in\mathbf{\Gamma}} |n^{-\gamma_*-1}\mathbf{G}(\gamma_*)'\mathbf{MT}(\gamma) - \beta_* g(\gamma_*,\gamma)| = o_p(1)$.

Therefore, using these two facts, it follows that

$$QLR_n = \sup_{\gamma \in \mathbf{\Gamma}} \frac{\{(n^{-1/2}\mathbf{U} + n^{-\gamma_*-1}\mathbf{G}(\gamma_*))'\mathbf{MT}(\gamma)\}^2}{\widehat{\sigma}_{n,0}^2\,(n^{-1}\mathbf{T}(\gamma)'\mathbf{MT}(\gamma))} \Rightarrow \sup_{\gamma\in\mathbf{\Gamma}} \left\{\frac{\widetilde{\mathcal{G}}(\gamma) + \beta_* g(\gamma_*,\gamma)}{\sigma_* g^{1/2}(\gamma,\gamma)}\right\}^2.$$

By the definitions of $\widetilde{\mathcal{Z}}(\cdot)$ and $\sigma^2(\cdot)$, viz., $\widetilde{\mathcal{Z}}(\cdot) := \widetilde{\mathcal{G}}(\cdot)/\{\sigma(\cdot)\}$ and $\sigma^2(\gamma) := \sigma_*^2 g(\gamma,\gamma)$, $QLR_n \Rightarrow \sup_{\gamma\in\mathbf{\Gamma}}(\widetilde{\mathcal{Z}}(\gamma) + \beta_* g(\gamma_*,\gamma)/\{\sigma_* g^{1/2}(\gamma,\gamma)\})^2$, as desired.

(*v*) By the definitions of the QLR test and $\varepsilon(n)$,

$$QLR_n = \sup_{\gamma \in \mathbf{\Gamma}} \frac{\{(n^{-1/2}\mathbf{U} + (n\varepsilon(n)m(n))^{-1}\mathbf{Q})'\mathbf{MT}(\gamma)\}^2}{\widehat{\sigma}_{n,0}^2\,(n^{-1}\mathbf{T}(\gamma)'\mathbf{MT}(\gamma))}.$$

As in the proof of (*iv*), we examine on the asymptotic behaviors of $\widehat{\sigma}_{n,0}^2$ and $(n\varepsilon(n)m(n))^{-1}\mathbf{Q})'\mathbf{MT}(\gamma)$ as the other terms are already examined.

First, we focus on the asymptotic behavior of $\widehat{\sigma}_{n,0}^2$. Note that

$$\begin{aligned}\widehat{\sigma}_{n,0}^2 &= n^{-1}(\mathbf{U} + n^{-1/2}(\varepsilon(n)m(n))^{-1}\mathbf{Q})'\mathbf{M}(\mathbf{U} + n^{-1/2}(\varepsilon(n)m(n))^{-1}\mathbf{Q})\\ &= n^{-1}\mathbf{U}'\mathbf{MU} + 2n^{-3/2}(\varepsilon(n)m(n))^{-1}\mathbf{Q}'\mathbf{MU} + n^{-2}(\varepsilon(n)m(n))^{-2}\mathbf{Q}'\mathbf{MQ} = \sigma_*^2 + o_p(1)\end{aligned}$$

by (A.24) and (A.25).

Second, we examine $(n\varepsilon(n)m(n))^{-1}\mathbf{Q})'\mathbf{M}\mathbf{T}(\gamma)$. Note that $(n\varepsilon(n)m(n))^{-1}\mathbf{Q})'\mathbf{M}\mathbf{T}(\cdot)$ uniformly converges to $p(\cdot)$ by (A.22).

We combine all these asymptotic behaviors of the terms that constitute the QLR test and obtain that

$$QLR_n = \sup_{\gamma\in\mathbf{\Gamma}} \frac{\{(n^{-1/2}\mathbf{U} + (n\varepsilon(n)m(n))^{-1}\mathbf{Q})'\mathbf{M}\mathbf{T}(\gamma)\}^2}{\widehat{\sigma}^2_{n,0}\,(n^{-1}\mathbf{T}(\gamma)'\mathbf{M}\mathbf{T}(\gamma))} \Rightarrow \sup_{\gamma\in\mathbf{\Gamma}} \left\{\widetilde{\mathcal{Z}}(\gamma) + \frac{p(\gamma)}{\sigma_* g^{1/2}(\gamma,\gamma)}\right\}^2$$

using the definition of $\widetilde{\mathcal{Z}}(\cdot)$. This completes the proof. ■

Remarks 2. *From the fact that* $n^{-\gamma_*-1}\mathbf{G}(\gamma_*)'\mathbf{M}\widetilde{\mathbf{L}}_1 = \beta_* n^{-1}\mathbf{T}(\gamma_*)'\mathbf{M}\widetilde{\mathbf{L}}_1$ *and* $n^{-\gamma_*-1}\mathbf{G}(\gamma_*)'\mathbf{M}\widetilde{\mathbf{C}}_1 = \beta_* n^{-1}\mathbf{T}(\gamma_*)'\mathbf{M}\widetilde{\mathbf{C}}_1$*, we now obtain that*

$$\frac{1}{n}\mathbf{T}(\gamma_*)'\mathbf{M}\widetilde{\mathbf{L}}_1 = -\frac{1}{(\gamma_*+1)^2} - \widetilde{\mathbf{A}}_{4,1}(\gamma_*)'(\widetilde{\mathbf{A}}_{1,1})^{-1}\widetilde{\mathbf{A}}_{2,1} + o_p(1), \quad \textit{and}$$

$$\frac{1}{n}\mathbf{T}(\gamma_*)'\mathbf{M}\widetilde{\mathbf{C}}_1 = -\frac{1}{(\gamma_*+2)^2} - \widetilde{\mathbf{A}}_{4,1}(\gamma_*)'(\widetilde{\mathbf{A}}_{1,1})^{-1}\widetilde{\mathbf{A}}_{3,1} + o_p(1)$$

*using Lemma A12(*i*). Therefore,*

$$\frac{1}{n^{\gamma_*+1}}\mathbf{G}(\gamma_*)'\mathbf{M}\widetilde{\mathbf{L}}_1 \stackrel{\mathbb{P}}{\to} \widetilde{\mu}_0 := -\frac{\beta_*}{(\gamma_*+1)^2} - \beta_*\widetilde{\mathbf{A}}_{4,1}(\gamma_*)'(\widetilde{\mathbf{A}}_{1,1})^{-1}\widetilde{\mathbf{A}}_{2,1}, \quad \textit{and}$$

$$\frac{1}{n^{\gamma_*+1}}\mathbf{G}(\gamma_*)'\mathbf{M}\widetilde{\mathbf{C}}_1 \stackrel{\mathbb{P}}{\to} \widetilde{\mu}_1 := -\frac{\beta_*}{(\gamma_*+2)^2} - \beta_*\widetilde{\mathbf{A}}_{4,1}(\gamma_*)'(\widetilde{\mathbf{A}}_{1,1})^{-1}\widetilde{\mathbf{A}}_{3,1}.$$

*We also showed in the proof of Theorem 6(*i*) that* $n^{-1}\widehat{\sigma}^2_{n,0}\widetilde{\mathbf{L}}'\mathbf{M}\widetilde{\mathbf{L}} \stackrel{\text{a.s.}}{\to} \widetilde{\sigma}^2_0 := \sigma^2_*(2 - \widetilde{\mathbf{A}}'_{2,1}\widetilde{\mathbf{A}}^{-1}_{1,1}\widetilde{\mathbf{A}}_{2,1})$ *and* $n^{-1}\widehat{\sigma}^2_{n,0}\widetilde{\mathbf{C}}'\mathbf{M}\widetilde{\mathbf{C}} \stackrel{\text{a.s.}}{\to} \widetilde{\sigma}^2_1 := \sigma^2_*(2/27 - \widetilde{\mathbf{A}}'_{3,1}\widetilde{\mathbf{A}}^{-1}_{1,1}\widetilde{\mathbf{A}}_{3,1})$*. Therefore,* $QLR_n^{(\gamma=0)} \stackrel{\text{A}}{\sim} (\widetilde{\mathcal{Z}}_0 + \widetilde{\mu}_0/\widetilde{\sigma}_0)^2$ *and* $QLR_n^{(\gamma=1)} \stackrel{\text{A}}{\sim} (\widetilde{\mathcal{Z}}_1 + \widetilde{\mu}_1/\widetilde{\sigma}_1)^2$ *under the same condition as in Theorem 6(iv), where* $\widetilde{\mathcal{Z}}_0 := \widetilde{\mathcal{G}}_0/\widetilde{\sigma}_0$ *and* $\widetilde{\mathcal{Z}}_1 := \widetilde{\mathcal{G}}_1/\widetilde{\sigma}_1$*.*

We also note that

$$\frac{1}{nm(n)\varepsilon(n)}\mathbf{Q}'\mathbf{M}\widetilde{\mathbf{L}}_1 = \frac{1}{n\varepsilon(n)}\mathbf{L}'\widetilde{\mathbf{L}}_1 - \frac{1}{n\varepsilon(n)}\mathbf{L}'\mathbf{Z}\left(\frac{1}{n}\mathbf{Z}\mathbf{Z}'\right)^{-1}\frac{1}{n}\mathbf{Z}'\widetilde{\mathbf{L}}_1, \quad \textit{and}$$

$$\frac{1}{nm(n)\varepsilon(n)}\mathbf{Q}'\mathbf{M}\widetilde{\mathbf{C}}_1 = \frac{1}{n\varepsilon(n)}\mathbf{L}'\widetilde{\mathbf{C}}_1 - \frac{1}{n\varepsilon(n)}\mathbf{L}'\mathbf{Z}\left(\frac{1}{n}\mathbf{Z}\mathbf{Z}'\right)^{-1}\frac{1}{n}\mathbf{Z}'\widetilde{\mathbf{C}}_1.$$

Given these, we apply lemma 4.1 of Phillips (2007) and obtain that

$$\frac{1}{n\varepsilon(n)m(n)}\mathbf{Q}'\mathbf{M}\widetilde{\mathbf{L}}_1 = \ddot{\mu}_0 := 2 + \widetilde{\mathbf{A}}'_{5,1}(\widetilde{\mathbf{A}}_{1,1})^{-1}\widetilde{\mathbf{A}}_{2,1}, \quad \textit{and}$$

$$\frac{1}{n\varepsilon(n)m(n)}\mathbf{Q}'\mathbf{M}\widetilde{\mathbf{C}}_1 = \ddot{\mu}_1 := \frac{1}{4} + \widetilde{\mathbf{A}}'_{5,1}(\widetilde{\mathbf{A}}_{1,1})^{-1}\widetilde{\mathbf{A}}_{3,1}.$$

Therefore, $QLR_n^{(\gamma=0)} \overset{\text{A}}{\sim} (\widetilde{\mathcal{Z}}_0 + \ddot{\mu}_0/\widetilde{\sigma}_0)^2$ *and* $QLR_n^{(\gamma=1)} \overset{\text{A}}{\sim} (\widetilde{\mathcal{Z}}_1 + \ddot{\mu}_1/\widetilde{\sigma}_1)^2$ *under the same condition as in Theorem 6(v).* □

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