Strictly Increasing Noncentral Correlation Coefficient between Powered Positive Variables

JIN SEO CHO

School of Economics

Yonsei University

50 Yonsei-ro, Seodaemun-gu, Seoul, 120-749, Korea

Email: jinseocho@yonsei.ac.kr

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Abstract

The noncentral correlation coefficient between two variables X and Y is defined as $\rho(X,Y) := E[XY]/\{E[X^2]^{1/2}E[Y^2]^{1/2}\}$. We show that $\rho(X^s, X^{s_*})$ is a strictly increasing function of s on $(0, s_*)$, where X is a positive variable. A couple of examples are provided as applications of this property.

Key Words: Noncentral correlation coefficient; Positive variable.

JEL Classification: C12, C18, C46, C52.

1 Introduction

Data analysis using noncentral correlation coefficient is popular in regression analysis if the intercept term is absent in regression equations. It is particularly so because we can represent regression coefficients as functions of noncentral correlation coefficients.

In this study, we examine the noncentral correlation coefficient between X^s and X^{s_*}, where X is a positive variable and show that it is a strictly increasing function of s on $(1, s_*)$. This property is associated with a couple of examples: (i) an inequality property on the products of two Gamma functions and (ii) the power of the quasi-likelihood ratio (QLR) test statistic that is examined by Cho and Ishida (2012) to test for neglected nonlinearity using the power transform of regressors.

The plan of this study is as follows. In Section 2, we show that the noncentral correlation coefficient is a strictly increasing function and provide its applications. Concluding remarks are provided in Section 3.

2 Strictly Increasing Noncentral Correlation Coefficient and Applications

We let the noncentral correlation coefficient between X and Y be

$$\rho(X,Y) := \frac{E[X \cdot Y]}{E[X^2]^{1/2} E[Y^2]^{1/2}}.$$

We show that $\rho(X^{(\cdot)}, X^{s_*})$ is strictly increasing on $(0, s_*)$, where X is a positive variable. The following lemma is useful for this purpose.

Lemma 1. *When W is positive, for any u, $v > 0$, $\mathbb{E}[W^{u+v}\log(W)]\mathbb{E}[W^v] \geq \mathbb{E}[W^{u+v}]\,\mathbb{E}[\log(W)W^v]$.* □

Proof of Lemma 1: Note that $w > z$ if and only if $w^u > z^u$ or $\log(w) > \log(z)$. Therefore, $\int\int(w^u - z^u)(\log(w) - \log(z))w^v z^v d\mathbb{P}(w)d\mathbb{P}(z) \geq 0$, where $\mathbb{P}$ is the probability measure associated with W. That is, $2\mathbb{E}[W^{u+v}\log\,(W)]\mathbb{E}[W^v] - 2\mathbb{E}[W^{u+v}]\mathbb{E}[W^v\log(W)] \geq 0$, as desired. ■

The proof of Lemma 1 can be thought of as an application of Fortuin, Kasteleyn, and Ginibre's (1971) inequality. If the equality holds in Lemma 1, W is constant almost surely.

The main theorem of this study is now provided.

Theorem 2. *Given that X is a positive variable such that for any c, $\mathbb{P}(X = c) < 1$, if for some $s_* > 0$, $\mathbb{E}[X^{2s_*}] < \infty$ and $\mathbb{E}[|X^{2s_*}\log(X)|] < \infty$, $\rho(X^{(\cdot)}, X^{s_*})$ is a strictly increasing function on $(0, s_*)$.* □

Proof of Theorem 2: Let $f(s) := \rho(X^{(\cdot)}, X^{s_*})$ and note that

$$f'(s) = \frac{1}{\mathbb{E}[X^{2s}]^{3/2}\mathbb{E}[X^{s_*}]]^{1/2}}\left(\mathbb{E}[X^{s+s_*}\log(X^{2s})]\mathbb{E}[X^{2s}] - \mathbb{E}[X^{s+s_*}]\mathbb{E}[X^{2s}\log(X^{2s})]\right),$$

and the denominator is always greater than zero. If we let X^{2s}, $(s_* - s)/(2s)$, and 1 be W, u, and v of Lemma 1, respectively, the numerator is $\mathbb{E}[W^{u+1}\log(W)]\mathbb{E}[W] - \mathbb{E}[W^{u+1}]\mathbb{E}[W\log(W)]$, which is strictly greater than zero as given in Lemma 1 and the fact that X is not constant. This completes the proof. ■

We now provide two applications of Theorem 2.

Example 1 (Inequality between the products of Gamma Functions)**.** Suppose that Y follows a Gamma distribution: for some $m_* > 0$, the probability density function of Y is $f(y) = y^{m_*-1}\mathrm{e}^{-y}/\Gamma(m_*)$. Then,

$\mathbb{E}[Y^{s+s_*}] = \Gamma(s+s_*+m_*)/\Gamma(m_*)$, so that Theorem 2 implies that for any $s < s' < s_*$,

$$\frac{\Gamma(s+s_*+m_*)}{\Gamma(s'+s_*+m_*)} < \left(\frac{\Gamma(2s+m_*)}{\Gamma(2s'+m_*)}\right)^{1/2}.$$

If we let $a := s+s_*+m_*$, $b := s'-s$, and $c := s_*-s$, this inequality can also be stated as follows: for any a, b, and $c > 0$ such that $a > c > b$,

$$\frac{\Gamma(a)}{\Gamma(a+b)} < \left(\frac{\Gamma(a-c)}{\Gamma(a+2b-c)}\right)^{1/2}.$$

Note that the right side is strictly increasing with respect to c, so that the minimal upper bound of the left side is obtained by letting c converge to b, implying that $\Gamma(a)^2 < \Gamma(a+b)\Gamma(a-b)$. This is the same inequality as (3.13) of Dragomir, Agarwal, and Barnett (2000). □

Example 2 (Power of the QLR Test Statistic)**.** Cho and Ishida (2012) consider the power of the QLR statistic that tests the linearity assumption of $\mathbb{E}[Y|X]$ with respect to X using the power transform of X. The QLR test statistic is defined as

$$QLR_n := n\left(1 - \frac{\widehat{\sigma}^2_{n,1}}{\widehat{\sigma}^2_{n,0}}\right),$$

where $\widehat{\sigma}^2_{n,0}$ and $\widehat{\sigma}^2_{n,1}$ are estimated variances of projection errors under the null and alternative hypotheses, respectively.

We examine the QLR test statistic using Theorem 2 and obtain its power property using the essential model part of Cho and Ishida (2012), that investigate only the asymptotic null distribution. We specifically suppose that $\mathbb{E}[Y|X] = \beta_* X^{s_*}$, where X is a positive variable with $s_* > 0$ and $\beta_* \neq 0$. If $s_* = 1$, the linearity of $\mathbb{E}[Y|X]$ holds with respect to X, and we identify this condition as the null hypothesis as in Cho and Ishida (2012). Therefore, $\widehat{\sigma}^2_{n,0} := n^{-1}\sum_{t=1}^n (Y_t - \widehat{\beta}_{n,0}X_t)^2$ and $\widehat{\sigma}^2_{n,1} := n^{-1}\sum_{t=1}^n (Y_t - \widehat{\beta}_{n,1}X_t^{\widehat{s}_{n,1}})^2$, where $\widehat{\beta}_{n,0}$ and $(\widehat{\beta}_{n,1}, \widehat{s}_{n,1})$ are the least squares estimators obtained under the null and alternative model assumptions, respectively. Note that $\sigma(s)^2 := \min_\beta \mathbb{E}[(Y - \beta X^s)^2] = \mathbb{E}[U^2] + \min_\beta \mathbb{E}[(\beta_* X^{s_*} - \beta X^s)^2] = \mathbb{E}[U^2] + \beta_*^2\mathbb{E}[X^{2s_*}] - \beta_*^2\mathbb{E}[X^{2s}]^{-1}\mathbb{E}[X^{s_*+s}]$, where $U \equiv Y - \mathbb{E}[Y|X]$, implying that for $s', s'' < s_*$, $\sigma(s'')^2 \leq \sigma(s')^2$ if and only if $\mathbb{E}[X^{2s'}]^{-1}\mathbb{E}[X^{s_*+s'}] \leq \mathbb{E}[X^{2s''}]^{-1}\mathbb{E}[X^{s_*+s''}]$. We also note that $\mathbb{E}[X^{2s}]^{-1}\mathbb{E}[X^{s_*+s}] \equiv \rho(X^s, X^{s_*})^2\mathbb{E}[X^{2s_*}]$. Therefore, $\sigma(s'')^2 \leq \sigma(s')^2$ if and only if $\rho(X^{s'}, X^{s_*})^2 \leq \rho(X^{s''}, X^{s_*})^2$. For any $s > 0$, $\rho(X^s, X^{s_*}) > 0$, so that if $s' < s''$, $\sigma(s'')^2 < \sigma(s')^2$. Now, suppose that $s' = 1$ and let $s'' := \arg\min_{s\in S}\sigma(s)^2$, where S is a parameter space for s_* under the alternative model assumption. We suppose that S includes 1 but may or may not include s_*. Then, applying the law of large numbers yields that QLR_n/n is asymptotically equal to $1 - \sigma(s'')^2/\sigma(1)^2$ that is in $(0,1)$.

Even though S does not include s_*, the QLR test statistic has an asymptotic power from this.

This power property is similar to that in Baek, Cho, and Phillips (theorem 6(ii), 2015) that assume a model for time trend by its power transform under the nonstationary data assumption: $X_t = t$, and they show that the QLR test statistic has an asymptotic power and a convergence rate that depends on whether $s_* \in S$ or not. □

3 Concluding Remarks

In this study, we examined the noncentral correlation coefficient between X^s and X^{s_*}, where X is positive and showed that the coefficient is a strictly increasing function of s. Two applications of this property are provided.

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