An Alternative Proof that OLS is BLUE

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Abstract

We provide an alternative proof that the Ordinary Least Squares estimator is the (conditionally) best linear unbiased estimator.

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Textbook proofs that OLS is BLUE are often somewhat lengthy, or require in-depth knowledge of matrix algebra (*e.g.,* Judge, *et al.*, 1988, Davidson and McKinnon, 2003, and Wooldridge, 2009, Appendix E). Here we give an alternative proof that is relatively short and assumes only the rudimentary matrix algebra. We impose the next conditions by following Wooldridge (2009, Appendix E),

Assumptions *1. For* n, $k \in \mathbb{N}^+$, $\mathbf{Y} \in \mathbb{R}^n$ *and* $\mathbf{X} \in \mathbb{R}^{n \times k}$ *such that* $\mathbf{X}$ *has rank* k, *and* $\mathbf{U} := \mathbf{Y} - E[\mathbf{Y}|\mathbf{X}]$.

2. For some $\boldsymbol{\beta}_* \in \mathbb{R}^k$, $E[\mathbf{Y}|\mathbf{X}] = \mathbf{X}\boldsymbol{\beta}_*$.

3. For some $\sigma_*^2 > 0$, $\mathrm{var}[\mathbf{U}|\mathbf{X}] = \sigma_*^2 \mathbf{I}_n$.

The following facts are standard. The linear estimators of $\boldsymbol{\beta}_*$ have the form $\tilde{\boldsymbol{\beta}}_n := \mathbf{B}\mathbf{Y}$, where $\mathbf{B} \in \mathbb{R}^{k \times n}$ and defined as a set of functions of $\mathbf{X}$. Given linearity, A.1, and A.2, $E(\tilde{\boldsymbol{\beta}}_n|\mathbf{X}) = \boldsymbol{\beta}_*$, i.e., $\tilde{\boldsymbol{\beta}}_n$ is (conditionally) unbiased, if and only if $\mathbf{B}\mathbf{X} = \mathbf{I}_k$. In what follows, $\tilde{\boldsymbol{\beta}}_n$ is linear unbiased. Given

A.1, the OLS estimator $\hat{\boldsymbol{\beta}}_n := (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ exists and is unique; it is a linear estimator of $\boldsymbol{\beta}_*$ with $\mathbf{A} := (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Given A.1 and A.2, $\hat{\boldsymbol{\beta}}_n$ is linear unbiased.

Observe that for all $\mathbf{B}$ such that $\mathbf{B}\mathbf{X} = \mathbf{I}_k$,

$$\mathbf{A}\mathbf{B}' = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{B}' = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{I}_k = \mathbf{A}\mathbf{A}' = \mathbf{B}\mathbf{A}'.$$

Given A.1 – A.3, $\mathrm{cov}(\tilde{\boldsymbol{\beta}}_n|\mathbf{X}) := E([\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_*][\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_*]'|\mathbf{X}) = E(\mathbf{B}\mathbf{U}\mathbf{U}'\mathbf{B}'|\mathbf{X}) = \sigma_*^2\mathbf{B}\mathbf{B}'$, as $E(\mathbf{U}\mathbf{U}'|\mathbf{X}) = \sigma_*^2\mathbf{I}_n$. Also, it trivially follows that $\mathrm{cov}(\hat{\boldsymbol{\beta}}_n|\mathbf{X}) = \sigma_*^2\mathbf{A}\mathbf{A}' = \sigma_*^2(\mathbf{X}'\mathbf{X})^{-1}$.

With this foundation, we have

Theorem. *Given A.1 – A.3, $\hat{\boldsymbol{\beta}}_n$ is the best linear unbiased estimator. That is, for all $\mathbf{X}$ and $\mathbf{B}$ such that $\mathbf{B}\mathbf{X} = \mathbf{I}_k$, $\mathrm{cov}(\tilde{\boldsymbol{\beta}}_n|\mathbf{X}) - \mathrm{cov}(\hat{\boldsymbol{\beta}}_n|\mathbf{X}) = \sigma_*^2[\mathbf{B}\mathbf{B}' - \mathbf{A}\mathbf{A}']$ is positive semi-definite (psd).*

Proof: We show that $\mathbf{B}\mathbf{B}' - \mathbf{A}\mathbf{A}'$ is psd. As $\mathbf{A}\mathbf{B}' = \mathbf{A}\mathbf{A}'$,

$$\mathbf{B}\mathbf{B}' - \mathbf{A}\mathbf{A}' = \mathbf{B}\mathbf{B}' - \mathbf{A}\mathbf{B}'.$$

As $\mathbf{B}\mathbf{A}' = \mathbf{A}\mathbf{A}'$,
$$\mathbf{B}\mathbf{B}' - \mathbf{A}\mathbf{A}' = \mathbf{B}\mathbf{B}' - \mathbf{A}\mathbf{B}' - \mathbf{B}\mathbf{A}' + \mathbf{A}\mathbf{A}'.$$

Collecting terms gives
$$\mathbf{B}\mathbf{B}' - \mathbf{A}\mathbf{A}' = [\mathbf{B} - \mathbf{A}][\mathbf{B} - \mathbf{A}]',$$

a positive semi-definite matrix. ∎

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