Notations in "Testing the Equality of Two Positive-Definite Matrices with Application to Information Matrix Testing"

JIN SEO CHO
School of Economics
Yonsei University
50 Yonsei-ro, Seodaemun-gu, Seoul 120-749, Korea
Email: jinseocho@yonsei.ac.kr

HALBERT WHITE
Department of Economics
University of California, San Diego
9500 Gilman Dr., La Jolla, CA, 92093-0508, U.S.A.
Email: hwhite@weber.ucsd.edu

First version: May 2014 This version: May 2014

Abstract

We collect the notations defined in "Testing the Equality of Two Positive-Definite Matrices with Application to Information Matrix Testing" by Cho and White (2014).

Objects to be Estimated

- $\mathbf{A}_* := \mathbf{A}(\boldsymbol{\theta}_*)$.
- $\mathbf{B}_* := \mathbf{B}(\boldsymbol{\theta}_*)$.
- $\mathbf{D}_* := \mathbf{B}_*\mathbf{A}_*^{-1}$.
- $\Sigma_{\mathbf{b},*} := E[\mathbf{b}(Y_t, \mathbf{X}_t)\mathbf{b}(Y_t, \mathbf{X}_t)']$.
- $T_* := k^{-1}\mathrm{tr}[\mathbf{D}_*] - 1$.
- $D_* := \det[\mathbf{D}_*]^{1/k} - 1$.
- $\boldsymbol{\mu}_{\mathbf{a},*} := E[\mathbf{a}(Y_t, \mathbf{X}_t)]$.
- $\Sigma_{\mathbf{a},*} := E[\mathbf{a}(Y_t, \mathbf{X}_t)\mathbf{a}(Y_t, \mathbf{X}_t)']$.
- $\boldsymbol{\mu}_{\mathbf{b},*} := E[\mathbf{b}(Y_t, \mathbf{X}_t)]$.
- $\bar{\mathbf{B}}_* := \bar{\mathbf{B}}(\boldsymbol{\theta}_*)$.
- $\bar{\mathbf{A}}_* := \bar{\mathbf{A}}(\boldsymbol{\theta}_*)$.

Estimators

- $\mathbf{A}_n := \mathbf{A}_n(\boldsymbol{\theta}_*)$.
- $\mathbf{B}_n := \mathbf{B}_n(\boldsymbol{\theta}_*)$.
- $\mathbf{D}_n := \mathbf{B}_n\mathbf{A}_n^{-1}$.
- $\widehat{\mathbf{A}}_n := \mathbf{A}_n(\widehat{\boldsymbol{\theta}}_n)$.
- $\widehat{\mathbf{B}}_n := \mathbf{B}_n(\widehat{\boldsymbol{\theta}}_n)$.
- $\widehat{\mathbf{D}}_n := \widehat{\mathbf{B}}_n\widehat{\mathbf{A}}_n^{-1}$.

- $\widetilde{\mathbf{A}}_n := \widehat{\Sigma}_{\mathbf{a},n} - \widehat{\boldsymbol{\mu}}_{\mathbf{a},n}\widehat{\boldsymbol{\mu}}'_{\mathbf{a},n}$.
- $\widetilde{\mathbf{B}}_n := \widehat{\Sigma}_{\mathbf{b},n} - \widehat{\boldsymbol{\mu}}_{\mathbf{b},n}\widehat{\boldsymbol{\mu}}'_{\mathbf{b},n}$.
- $\widehat{\boldsymbol{\mu}}_{\mathbf{a},n} := n^{-1}\sum_{t=1}^{n}\mathbf{a}(Y_t,\mathbf{X}_t)$.
- $\widehat{\Sigma}_{\mathbf{a},n} := n^{-1}\sum_{t=1}^{n}\mathbf{a}(Y_t,\mathbf{X}_t)\mathbf{a}(Y_t,\mathbf{X}_t)'$.
- $\widehat{\boldsymbol{\mu}}_{\mathbf{b},n} := n^{-1}\sum_{t=1}^{n}\mathbf{b}(Y_t,\mathbf{X}_t)$.
- $\widehat{\Sigma}_{\mathbf{b},n} := n^{-1}\sum_{t=1}^{n}\mathbf{b}(Y_t,\mathbf{X}_t)\mathbf{b}(Y_t,\mathbf{X}_t)'$.
- $\widetilde{\mathbf{D}}_n := \widetilde{\mathbf{B}}_n\widetilde{\mathbf{A}}_n^{-1}$.
- $\widehat{Q}_n := \ln[\mathrm{tr}[\widehat{\mathbf{D}}_n]/k]$.
- $\widehat{L}_n := \ln[\det(\widehat{\mathbf{D}}_n)]/k$.

Test Bases

- $T_n := \mathrm{tr}[\mathbf{D}_n]/k - 1$.
- $D_n := \det[\mathbf{D}_n]^{1/k} - 1$.
- $S_n := \mathrm{tr}[\mathbf{D}_n]/k - \det[\mathbf{D}_n]^{1/k}$.
- $\widehat{T}_n := \mathrm{tr}[\widehat{\mathbf{D}}_n]/k - 1$.
- $\widehat{D}_n := \det[\widehat{\mathbf{D}}_n]^{1/k} - 1$.
- $\widehat{S}_n := \mathrm{tr}[\widehat{\mathbf{D}}_n]/k - \det[\widehat{\mathbf{D}}_n]^{1/k}$.
- $\widetilde{T}_n := \mathrm{tr}[\widetilde{\mathbf{D}}_n]/k - 1$.
- $\widetilde{D}_n := \det[\widetilde{\mathbf{D}}_n]^{1/k} - 1$.
- $\widetilde{S}_n := \mathrm{tr}[\widetilde{\mathbf{D}}_n]/k - \det[\widetilde{\mathbf{D}}_n]^{1/k}$.
- $\widehat{W}_n := \widehat{Q}_n - \widehat{L}_n$.
- $\widehat{M}_n := \widehat{T}_n - \widehat{L}_n$.

Operators

- for $j = 1,\ldots,\ell$, $\partial_j := (\partial/\partial\theta_j)$.
- for $i,j = 1,2,\ldots,\ell$, $\partial^2_{ji} := (\partial^2/\partial\theta_j\partial\theta_i)$.

Test Statistics

- $\mathscr{B}_n^{(1)} := nk^2\left(\frac{1}{2}T_n^2 + \frac{1}{2}D_n^2\right)$.
- $\mathscr{B}_n^{(2)} := 2nk\left(\frac{1}{2}T_n^2 + S_n\right)$.
- $\mathscr{B}_n^{(3)} := 2nk\left(\frac{1}{2}D_n^2 + S_n\right)$.
- $\widehat{\mathscr{B}}_n^{(1)} := nk^2\left(\frac{1}{2}\widehat{T}_n^2 + \frac{1}{2}\widehat{D}_n^2\right)$.
- $\widehat{\mathscr{B}}_n^{(2)} := 2nk\left(\frac{1}{2}\widehat{T}_n^2 + \widehat{S}_n\right)$.
- $\widehat{\mathscr{B}}_n^{(3)} := 2nk\left(\frac{1}{2}\widehat{D}_n^2 + \widehat{S}_n\right)$.
- $\widetilde{\mathscr{B}}_n^{(1)} := nk^2\left(\frac{1}{2}\widetilde{T}_n^2 + \frac{1}{2}\widetilde{D}_n^2\right)$.
- $\widetilde{\mathscr{B}}_n^{(2)} := 2nk\left(\frac{1}{2}\widetilde{T}_n^2 + \widetilde{S}_n\right)$.
- $\widetilde{\mathscr{B}}_n^{(3)} := 2nk\left(\frac{1}{2}\widetilde{D}_n^2 + \widetilde{S}_n\right)$.
- $\mathscr{L}\mathscr{R}_n^{(1)} := 2\{\ln[L_n(\widehat{\boldsymbol{\theta}}_n,\widehat{\mathbf{B}}_n)/L_n(\widehat{\boldsymbol{\theta}}_n,\widehat{\mathbf{A}}_n)]\}$.
- $\mathscr{L}\mathscr{R}_n^{(2)} := 2\{\ln[L_n(\widehat{\boldsymbol{\theta}}_n,\widehat{\mathbf{B}}_n)/L_n(\widetilde{\boldsymbol{\theta}}_n,\widetilde{d}_n\widehat{\mathbf{A}}_n)]\}$.
- $\mathscr{L}\mathscr{R}_n^{(3)} := 2\{\ln[L_n(\widetilde{\boldsymbol{\theta}}_n,\widetilde{d}_n\widehat{\mathbf{A}}_n)/L_n(\widetilde{\boldsymbol{\theta}}_n,\widehat{\mathbf{A}}_n)]\}$.

Supplementary Notations

- $\mathbf{M}_n := \mathbf{A}_*^{-1}(\mathbf{B}_n - \mathbf{A}_n)$.
- $\mathbf{S}_{j,*} := \mathbf{A}_*^{-1}(\partial_j\mathbf{B}_* - \partial_j\mathbf{A}_*)$.
- $\mathbf{K}_n := \mathbf{M}_n + \sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})\mathbf{S}_{j,*}$.
- $\mathbf{G}_{j,n} := \mathbf{B}_*^{-1}\partial_j(\mathbf{B}_n - \mathbf{B}_*)$.
- $\mathbf{H}_{j,n} := \mathbf{A}_*^{-1}\partial_j(\mathbf{A}_n - \mathbf{A}_*)$.
- $\mathbf{J}_{j,n} := \mathbf{G}_{j,n} - \mathbf{H}_{j,n}$.

- $\mathbf{W}_n := \mathbf{B}_*^{-1}(\mathbf{B}_n - \mathbf{B}_*)$.
- $\mathbf{U}_n := \mathbf{A}_*^{-1}(\mathbf{A}_n - \mathbf{A}_*)$.
- $\mathbf{P}_n := \mathbf{W}_n - \mathbf{U}_n$.
- $\mathbf{R}_{j,*} := \mathbf{B}_*^{-1}\partial_j \mathbf{B}_* - \mathbf{A}_*^{-1}\partial_j \mathbf{A}_*$.
- $\mathbf{L}_n := \mathbf{P}_n + \sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})\mathbf{R}_{j,*}$.
- $\mathbf{W}_{o,n} := \mathbf{B}_*^{-1}(\mathbf{B}_n - \mathbf{B}_{*,n})$.
- $\mathbf{M}_{o,n} := \mathbf{W}_{o,n} - \mathbf{U}_n$.
- $\mathbf{K}_{o,n} := \mathbf{M}_{o,n} + \sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})\mathbf{S}_{j,*}$.
- $\mathbf{G}_{j,o,n} := \mathbf{B}_*^{-1}\partial_j(\mathbf{B}_n - \mathbf{B}_{*,n})$.
- $\mathbf{J}_{j,o,n} := \mathbf{G}_{j,o,n} - \mathbf{H}_{j,n}$.
- $\mathbf{N}_* := \mathbf{B}_*^{-1}\bar{\mathbf{B}}_*$.
- $\mathbf{C}_{j,*} := \mathbf{B}_*^{-1}\partial_j \bar{\mathbf{B}}_* - \mathbf{N}_*\mathbf{B}_*^{-1}(\partial_j \mathbf{B}_*)$.
- $\widetilde{\mathbf{M}}_{o,n} := \mathbf{A}_*^{-1}(\widetilde{\mathbf{B}}_n - \mathbf{B}_{*,n}) - \mathbf{A}_*^{-1}(\widetilde{\mathbf{A}}_n - \mathbf{A}_*)$.
- $\widetilde{\mathbf{M}}_n := \mathbf{A}_*^{-1}(\widetilde{\mathbf{B}}_n - \widetilde{\mathbf{A}}_n)$.
- $\widetilde{\mathbf{W}}_n := \mathbf{B}_{*,n}^{-1}(\mathbf{B}_n - \mathbf{B}_{*,n})$.
- $\widetilde{\mathbf{P}}_n := \widetilde{\mathbf{W}}_n - \mathbf{A}_*^{-1}(\mathbf{A}_n - \mathbf{A}_*)$.
- $\widetilde{\mathbf{R}}_{j,*,n} := \mathbf{B}_{*,n}^{-1}\partial_j \mathbf{B}_{*,n} - \mathbf{A}_*^{-1}\partial_j \mathbf{A}_*$.
- $\widetilde{\mathbf{L}}_n := \widetilde{\mathbf{P}}_n + \sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})\widetilde{\mathbf{R}}_{j,*,n}$.
- $\widetilde{\mathbf{G}}_{j,n} := \mathbf{B}_{*,n}^{-1}\partial_j(\mathbf{B}_n - \mathbf{B}_{*,n})$.
- $\widetilde{\mathbf{J}}_{j,n} := \widetilde{\mathbf{G}}_{j,n} - \mathbf{H}_{j,n}$.
- $\widetilde{\boldsymbol{\theta}}_n := \widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*$.
- $\mathbf{Q}_n := (\mathbf{A}_*^{-1}\mathbf{B}_* - \det[\mathbf{D}_*]^{\frac{1}{k}}\mathbf{I})$.

Expansions

- $\widehat{T}_n^\star := \widehat{T}_{n,1}^\star + \widehat{T}_{n,2}^\star$.
- $\widehat{T}_{n,1}^\star := \frac{1}{k}\mathrm{tr}[\mathbf{K}_n]$.
- $\widehat{T}_{n,2}^\star := -\frac{1}{k}\mathrm{tr}[\mathbf{K}_n\mathbf{U}_n] + \frac{1}{k}[\mathrm{tr}[\mathbf{J}_{j,n} - \mathbf{M}_n\mathbf{A}_*^{-1}\partial_j\mathbf{A}_*]]'(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*) + \frac{1}{2k}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)'\nabla_{\boldsymbol{\theta}}^2\mathrm{tr}[\mathbf{D}_*](\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)$.
- $\widehat{D}_n^\star := \widehat{D}_{n,1}^\star + \widehat{D}_{n,2}^\star$.
- $\widehat{D}_{n,1}^\star := \frac{1}{k}\mathrm{tr}[\mathbf{K}_n]$.
- $$\widehat{D}_{n,2}^\star := \frac{1}{2k}\left(\frac{1}{k}-1\right)\mathrm{tr}[\mathbf{K}_n]^2 + \frac{1}{2k}(\mathrm{tr}[\mathbf{M}_n]^2 + \mathrm{tr}[\mathbf{U}_n^2] - \mathrm{tr}[\mathbf{W}_n^2]) + \frac{1}{k}[\mathrm{tr}[\mathbf{M}_n]\mathrm{tr}[\mathbf{S}_{j,*}]]'(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*) + \frac{1}{k}[\mathrm{tr}[\mathbf{J}_{j,n} + \mathbf{U}_n\mathbf{A}_*^{-1}\partial_j\mathbf{A}_* - \mathbf{W}_n\mathbf{A}_*^{-1}\partial_j\mathbf{B}_*]]'(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*) + \frac{1}{2k}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)'\nabla_{\boldsymbol{\theta}}^2\det[\mathbf{D}_*](\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*).$$
- $$\widehat{S}_n^\star := -\frac{1}{2k}\left(\frac{1}{k}-1\right)\mathrm{tr}[\mathbf{K}_n]^2 - \frac{1}{2k}(\mathrm{tr}[\mathbf{M}_n]^2 - \mathrm{tr}[\mathbf{M}_n^2]) + \frac{1}{k}\mathrm{tr}[\mathbf{M}_n\sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})\mathbf{S}_{j,*}] - \frac{1}{k}\mathrm{tr}[\mathbf{M}_n]\mathrm{tr}[\sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})\mathbf{S}_{j,*}] + \frac{1}{2k}\mathrm{tr}[(\sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})\mathbf{S}_{j,*})^2] - \frac{1}{2k}\mathrm{tr}[\sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})\mathbf{S}_{j,*}]^2.$$
- $\widehat{T}_{o,n} := \frac{1}{k}\mathrm{tr}[\mathbf{K}_{o,n}(\mathbf{I} - \mathbf{U}_n)] + \frac{1}{k}[\mathrm{tr}[\mathbf{J}_{j,o,n} - \mathbf{M}_{o,n}\mathbf{A}_*^{-1}\partial_j\mathbf{A}_*]]'(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*) + \frac{1}{2k}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)'\nabla_{\boldsymbol{\theta}}^2\mathrm{tr}[\mathbf{D}_*](\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)$.
- $$\widehat{D}_{o,n} := \frac{1}{k}\mathrm{tr}[\mathbf{K}_{o,n}] + \frac{1}{2k}\left(\frac{1}{k}-1\right)\mathrm{tr}[\mathbf{K}_{o,n}]^2 + \frac{1}{2k}(\mathrm{tr}[\mathbf{M}_{o,n}]^2 + \mathrm{tr}[\mathbf{U}_n^2] - \mathrm{tr}[\mathbf{W}_{o,n}^2]) + \frac{1}{k}[\mathrm{tr}[\mathbf{J}_{j,o,n} + \mathbf{U}_n\mathbf{A}_*^{-1}\partial_j\mathbf{A}_* - \mathbf{W}_{o,n}\mathbf{A}_*^{-1}\partial_j\mathbf{B}_*]]'(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*) + \frac{1}{k}[\mathrm{tr}[\mathbf{M}_{o,n}]\mathrm{tr}[\mathbf{S}_{j,*}]]'(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*) + \frac{1}{2k}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*)'\nabla_{\boldsymbol{\theta}}^2\det[\mathbf{D}_*](\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_*).$$

- $$\widehat{S}_{o,n} := -\frac{1}{2k}\left(\frac{1}{k}-1\right)\mathrm{tr}[\mathbf{K}_{o,n}]^2 - \frac{1}{2k}(\mathrm{tr}[\mathbf{M}_{o,n}]^2 - \mathrm{tr}[\mathbf{M}_{o,n}^2]) + \frac{1}{k}\mathrm{tr}[\mathbf{M}_{o,n}\sum_{j=1}^{\ell}(\widehat{\theta}_{j,n}-\theta_{j,*})\mathbf{S}_{j,*}]$$
$$-\frac{1}{k}\mathrm{tr}[\mathbf{M}_{o,n}]\mathrm{tr}[\sum_{j=1}^{\ell}(\widehat{\theta}_{j,n}-\theta_{j,*})\mathbf{S}_{j,*}] + \frac{1}{2k}\mathrm{tr}[(\sum_{j=1}^{\ell}(\widehat{\theta}_{j,n}-\theta_{j,*})\mathbf{S}_{j,*})^2] - \frac{1}{2k}\mathrm{tr}[\sum_{j=1}^{\ell}(\widehat{\theta}_{j,n}-\theta_{j,*})\mathbf{S}_{j,*}]^2.$$

- $\alpha_n := \alpha_n^{(1)} + \alpha_n^{(2)}$.
- $\alpha_n^{(1)} := \frac{1}{k}\mathrm{tr}[\mathbf{Q}_n\mathbf{L}_n]$.
- $\alpha_n^{(2)} := \frac{1}{k}[\mathrm{tr}[\mathbf{Q}_n\mathbf{J}_{j,n}]]'\widetilde{\boldsymbol{\theta}}_n - \frac{1}{k}\widetilde{\boldsymbol{\theta}}_n'[\mathrm{tr}[\mathbf{Q}_n\mathbf{R}_{j,*}\mathbf{A}_*^{-1}\partial_i\mathbf{A}_* + \mathbf{Q}_n(\mathbf{B}_*^{-1}\partial_{ji}^2\mathbf{B}_* - \mathbf{A}_*^{-1}\partial_{ji}^2\mathbf{A}_*)]]'\,\widetilde{\boldsymbol{\theta}}_n$.
- $$\beta_n := -\frac{1}{k}\mathrm{tr}[\mathbf{A}_*^{-1}\mathbf{B}_*\mathbf{L}_n\mathbf{U}_n] - \det[\mathbf{D}_*]^{\frac{1}{k}}\{\frac{1}{2k^2}\mathrm{tr}[\mathbf{L}_n]^2 - \frac{1}{2k}\mathrm{tr}[\mathbf{W}_n^2 - \mathbf{U}_n^2]\} - \frac{1}{k}[\mathrm{tr}[\mathbf{A}_*^{-1}\mathbf{B}_*\mathbf{P}_n\mathbf{A}_*^{-1}\partial_j\mathbf{A}_*]]'\widetilde{\boldsymbol{\theta}}_n$$
$$+\frac{1}{k}\det[\mathbf{D}_*]^{\frac{1}{k}}[\mathrm{tr}[\mathbf{U}_n\mathbf{R}_{j,*} + \mathbf{P}_n\mathbf{B}_*^{-1}\partial_j\mathbf{B}_*]]'\widetilde{\boldsymbol{\theta}}_n + \frac{1}{2k}\det[\mathbf{D}_*]^{\frac{1}{k}}\widetilde{\boldsymbol{\theta}}_n'[\mathrm{tr}[\mathbf{R}_{j,*}\mathbf{R}_{i,*}]]\widetilde{\boldsymbol{\theta}}_n.$$

References

CHO, J.S. AND WHITE, H. (2014): "Testing the Equality of Two Positive-Definite Matrices with Application to Information Matrix Testing," Discussion Paper, School of Economics, Yonsei University.