Technical Annex: Mathematical Proofs for "Quantile Cointegration in the Autoregressive Distributed-Lag Modelling Framework"

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August 2, 2014

Abstract

We provide mathematical proofs for main theorems and corollaries in "Quantile Cointegration in the Autoregressive Distributed-Lag Modelling Framework" by Cho, Kim, and Shin (2014).

JEL Classification: C22

Acknowledgements: We are mostly grateful to the Co-editor, Han Hong, and two anonymous referees for their helpful comments. We are also grateful to Youngsoo Bae, Stephane Bonhomme, Yongsung Chang, Kausik Chaudhuri, Seonghoon Cho, In Choi, Kyungwook Choi, Viet Anh Dang, Ana-Maria Fuertes, Matthew Greenwood-Nimmo, Hwankoo Kang, Dowan Kim, Junhan Kim, Gary Koop, Dongjin Lee, Viet Hoang Ngyuen, Sangsoo Park, Yangsoo Park, Peter Phillips, Kevin Reily, Xin Shen, Kyulee Shin, Peter Smith, Seungjoo Song, Peter Spencer, Till van Treeck, Mike Wickens, Ralf Wilke, seminar participants at the Bank of Korea, the Universities of Brunel, Korea, Leeds, Melbourne, Seoul, Sogang, Yonsei, York, Cass Business School, and the Institut für Makroökonomie und Konjunkturforschung (IMK, Dusseldorf), and conference delegates at the Conference in Honour of Professor PCB Phillips (University of York, 12-13 July 2012) for their helpful comments. Kim is grateful for financial support from the Yonsei University Research Fund (2011-1-0244). The usual disclaimer applies.

1 Introduction

This note provides mathematical proofs of the main theoretical results reported in Cho, Kim, and Shin (2014). We avoid possible confusions by using an equation number system different from that in Cho, Kim, and Shin (2014) using square brackets.

1 Proofs

We first provide a number of preliminary lemmas and a corollary that will be used in proving the main theorems.

Lemma A1. *Under Assumption 1,*

(i)$\sum_{t=1}^{n} \mathbf{W}_{t-i} = O_{\mathbb{P}}(\sqrt{n})$ *for* $i = 0, 1, \ldots$; *and*

(ii) for each τ, $\sum_{t=1}^{n} \bar{\mathbf{K}}_t(\tau) = O_{\mathbb{P}}(\sqrt{n})$ *and* $\sum_{t=1}^{n} \mathbf{K}_t(\tau) = O_{\mathbb{P}}(n)$. □

Proof of Lemma A1: By letting $r = 1$, Assumption 1(vi) implies that $n^{-1/2} \sum_{t=1}^{n} \bar{\mathbf{W}}_t \Rightarrow \boldsymbol{\mathcal{B}}_{\mathbf{W}}(1)$ and $n^{-1/2} \sum_{t=1}^{n} \bar{\mathbf{K}}_t(\tau) \Rightarrow \boldsymbol{\mathcal{B}}_{\mathbf{K}}(1, \tau)$. Furthermore, we can apply the ergodic theorem to $n^{-1} \sum_{t=1}^{n} \mathbf{K}_t(\tau) \to \mathbb{E}[\mathbf{K}_t(\tau)]$ in probability, which completes the proof. ■

Lemma A2. *Under Assumption 1,*

(i) $n^{-1} \sum_{t=1}^{n} \mathbf{W}_{t-i}\mathbf{W}'_{t-j} \to \mathbb{E}[\mathbf{W}_{t-i}\mathbf{W}'_{t-j}]$ almost surely (a.s.) *for* $i, j = 0, 1, \ldots, q-1$*;*

(ii) $\sum_{t=1}^{n} \mathbf{W}_{t-i}\mathbf{X}'_t = O_{\mathbb{P}}(n)$ *and* $\sum_{t=1}^{n} U_{t-i}(\tau)\mathbf{X}_t = O_{\mathbb{P}}(n^{3/2})$*, where* $i = 0, 1, \ldots$;

(iii) For each τ, $n^{-1} \sum_{t=1}^{n} \mathbf{K}_t(\tau)\mathbf{K}_t(\tau)' \to \mathbb{E}[\mathbf{K}_t(\tau)\mathbf{K}_t(\tau)']$ a.s.*;*

(iv) For each τ, $n^{-1} \sum_{t=1}^{n} \mathbf{K}_t(\tau)\mathbf{W}'_{t-i} \to \mathbb{E}[\mathbf{K}_t(\tau)\mathbf{W}'_{t-i}]$ a.s.*, where* $i = 0, 1, \ldots, q-1$*; and*

(v) For each τ, $n^{-3/2} \sum_{t=1}^{n} \mathbf{K}_t(\tau)\mathbf{X}'_t \Rightarrow \mathbb{E}[\mathbf{K}_t(\tau)] \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)'dr$. □

Proof of Lemma A2: (i) By applying the ergodic theorem to Assumption 1(ii), we obtain the result in Lemma A2(i).

(ii) We follow the proof of Proposition 18.1(d) in Hamilton (1994, pp. 562–563) by letting his $\mathbf{u}_{t-s}$ and $\boldsymbol{\xi}_{t-1}$ be our $\mathbf{W}_{t-i}$ and $\mathbf{X}_t$, respectively. We then apply Assumption 1(vi), and derive $\sum_{t=1}^{n} \mathbf{W}_{t-i}\mathbf{X}'_t = O_{\mathbb{P}}(n)$ by induction. By Assumption 1(vi), $n^{-1/2} \sum_{t=1}^{\lfloor n(\cdot) \rfloor} \mathbf{W}_t \Rightarrow \bar{\boldsymbol{\mathcal{B}}}_W(\cdot)$. Then, it is straightforward to show that $\sum_{t=1}^{n} U_{t-i}(\tau)\mathbf{X}_t = O_{\mathbb{P}}(n)$ by the continuous mapping theorem and Assumptions 1(i) and 1(ii). Furthermore, it is elementary to show that $\sum_{t=1}^{n} \mathbf{X}_t = O_{\mathbb{P}}(n^{3/2})$. From these, it follows that $\sum_{t=1}^{n} U_{t-i}(\tau)\mathbf{X}_t = O_{\mathbb{P}}(n^{3/2})$.

(iii) We first note that $\{K_{t,i}(\tau)\}$ is a stationary and ergodic process. First, consider when $i \leq q$. Then,

$$K_{t,i}(\tau) = -\sum_{j=q-1}^{\infty} \boldsymbol{\xi}_{0,j*}(\tau)'\mathbf{W}_{t-i-j} + \sum_{j=0}^{\infty} \rho_{j*}(\tau)U_{t-i-j}(\tau).$$

From the definition of $\rho_{j*}(\tau)$ and Assumption 1(v), $\{\sum_{j=0}^{\infty} \rho_{j*}(\tau)U_{t-i-j}(\tau)\}$ is stationary and ergodic. Assumption 1(v) and Theorem 4.4.1 of Brockwell and Davis (1991, p. 122) also imply that $\{\sum_{j=q-1}^{\infty} \boldsymbol{\xi}_{0,j*}(\tau)'$ $\mathbf{W}_{t-i-j}\}$ is stationary, which implies that $\{\sum_{j=q-1}^{\infty} \boldsymbol{\xi}_{0,j*}(\tau)'\mathbf{W}_{t-i-j}\}$ is ergodic by Theorem 3.35 of White (2001). Thus, $\{K_{t,i}(\tau)\}$ is stationary and ergodic when $i \leq q$. We next consider when $i > q$. Then,

$$K_{t,i}(\tau) = -\boldsymbol{\beta}_*(\tau)' \sum_{j=0}^{i-q-1} \mathbf{W}_{t-q-j} + \sum_{j=0}^{\infty} \boldsymbol{\pi}_{j*}(\tau)'\mathbf{W}_{t-i-j} + \sum_{j=0}^{\infty} \rho_{j*}(\tau)U_{t-i-j}(\tau),$$

and $\{\sum_{j=0}^{\infty} \boldsymbol{\pi}_{j*}(\tau)' \sum_{t=1}^{n} \mathbf{W}_{t-i-j} + \sum_{j=0}^{\infty} \rho_{j*}(\tau) \sum_{t=1}^{n} U_{t-i-j}(\tau)\}$ is stationary and ergodic by the same logic as above. Furthermore, $\{\sum_{j=0}^{i-q-1} \mathbf{W}_{t-q-j}\}$ is a sequence of finite sums of the stationary and ergodic processes, so that $\{K_{t,i}(\tau)\}$ is stationary and ergodic even when $i > q$.

Given this, Assumption 1(vi) implies that $\mathbb{E}[K_{t,i}(\tau)^2] < \infty$. By applying the ergodic theorem to the definition of $\mathbf{K}_t(\tau)$, it follows that $n^{-1} \sum_{t=1}^{n} K_{t,i}(\tau)K_{t,j}(\tau) \to \mathbb{E}[K_{t,i}(\tau)K_{t,j}(\tau)]$ *a.s.* for $i, j = 1, 2, \ldots, p$. This verifies Lemma A2(iii).

(iv) Since $\mathbf{K}_t(\tau)$ and $\mathbf{W}_{t-i}$ are stationary and ergodic processes by Assumption 1(ii), $\{\mathbf{K}_t(\tau)\mathbf{W}'_{t-i}\}$ is a stationary and ergodic process by Theorem 3.35 of White (2001). Moreover, by Assumption 1(vi), $\mathbb{E}[K_{t,j}(\tau)^2] < \infty$ and $\mathbb{E}[W_{t-j,\ell}^2] < \infty$ for $j = 1, \ldots, p$ and $\ell = 1, \ldots, k$. Hence, the result in Lemma A2(iv) follows from the ergodic theorem.

(v) First, we consider the case with $i \leq q$. Then,

$$\sum_{t=1}^{n} K_{t,i}(\tau)\mathbf{X}'_t = -\sum_{j=q-1}^{\infty} \boldsymbol{\xi}_{0,j*}(\tau)' \underbrace{\sum_{t=1}^{n} \mathbf{W}_{t-i-j}\mathbf{X}'_t}_{O_{\mathbb{P}}(n)} + \sum_{j=0}^{\infty} \rho_{j*}(\tau) \underbrace{\sum_{t=1}^{n} U_{t-i-j}(\tau)\mathbf{X}'_t}_{O_{\mathbb{P}}(n^{3/2})}.$$

We now have that $\sum_{t=1}^{n} \mathbf{W}_{t-i-j}\mathbf{X}_t = O_{\mathbb{P}}(n)$ and $\sum_{t=1}^{n} U_{t-i-j}(\tau)\mathbf{X}_t = O_{\mathbb{P}}(n^{3/2})$ by Lemma A2(ii). Therefore, Assumption 1(v) implies $\sum_{t=1}^{n} K_{t,i}(\tau)\mathbf{X}'_t = O_{\mathbb{P}}(n^{3/2})$. Furthermore, $U_{t-i-j}(\tau)$ and $\mathbf{X}_t$ are independent by the definition of $U_{t-i-j}(\tau)$. This implies that $\sum_{t=1}^{n}(U_{t-i-j}(\tau) - \mathbb{E}[U_{t-i-j}(\tau)])\mathbf{X}_t = O_{\mathbb{P}}(n)$ by theorem 17. 3 of Hamilton (1994, pp. 505–506). This implies that $n^{-3/2} \sum_{t=1}^{n} K_{t,i}(\tau)\mathbf{X}'_t = \sum_{j=0}^{\infty} \rho_{j*}(\tau)\mathbb{E}[U_{t-i-j}(\tau)]\, n^{-3/2} \sum_{t=1}^{n} \mathbf{X}'_t = \mathbb{E}[K_{t,i}(\tau)] \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)'dr$.

Next, consider the case with $i > q$. Then, Lemma A2(ii) allows that

$$\sum_{t=1}^{n} K_{t,i}(\tau)\mathbf{X}_t'$$
$$= -\boldsymbol{\beta}_*(\tau)' \sum_{j=0}^{i-q-1} \underbrace{\sum_{t=1}^{n} \mathbf{W}_{t-q-j}\mathbf{X}_t'}_{O_{\mathbb{P}}(n)} + \sum_{j=0}^{\infty} \boldsymbol{\pi}_{j*}(\tau)' \underbrace{\sum_{t=1}^{n} \mathbf{W}_{t-i-j}\mathbf{X}_t'}_{O_{\mathbb{P}}(n)} + \sum_{j=0}^{\infty} \rho_{j*}(\tau) \underbrace{\sum_{t=1}^{n} U_{t-i-j}(\tau)\mathbf{X}_t'}_{O_{\mathbb{P}}(n^{3/2})}.$$

By the same reason as above, $\sum_{t=1}^{n} \mathbf{W}_{t-q-j}\mathbf{X}_t' = O_{\mathbb{P}}(n)$, $\sum_{t=1}^{n} \mathbf{W}_{t-i-j}\mathbf{X}_t' = O_{\mathbb{P}}(n)$, and $\sum_{t=1}^{n} U_{t-i-j}(\tau)\mathbf{X}_t' = O_{\mathbb{P}}(n^{3/2})$. Thus, Assumption 1(v) implies that $\sum_{t=1}^{n} K_{t,i}(\tau)\mathbf{X}_t'$ is $O_{\mathbb{P}}(n^{3/2})$, and $n^{-3/2}\sum_{t=1}^{n} K_{t,i}(\tau)\mathbf{X}_t' = n^{-3/2}\sum_{j=0}^{\infty} \rho_{j*}(\tau)\sum_{t=1}^{n} U_{t-i-j}(\tau)\mathbf{X}_t' + o_{\mathbb{P}}(1)$. The rest of the proof is identical to the case when $i \leq q$. This completes the proof. ■

Lemma A3. *Under Assumption 1,*

$$\sum_{t=1}^{n} \begin{bmatrix} n^{-3/2}\mathbf{X}_t \\ n^{-2}\mathbf{X}_t\mathbf{X}_t' \\ n^{-1}\psi_\tau[U_t(\tau)]\mathbf{X}_t \end{bmatrix} \Rightarrow \begin{bmatrix} \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)dr \\ \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)\bar{\boldsymbol{\mathcal{B}}}_W(r)'dr \\ \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)d\mathcal{B}_\psi(r,\tau) \end{bmatrix}. \qquad \square$$

Proof of Lemma A3: By Assumption 1(vi), we have $n^{-1/2}\sum_{t=1}^{\lfloor n(\cdot)\rfloor} \mathbf{W}_t \Rightarrow \bar{\boldsymbol{\mathcal{B}}}_W(\cdot)$. Then, application of the continuous mapping theorem and Lemma 3.1(e) in Phillips and Durlauf (1986) delivers the desired result. ■

The following Corollary immediately follows from the previous Lemmas.

Corollary A1. *Under Assumption 1,*

(i) Let $\mathbf{D}_{\mathbf{G}} := \text{diag}([\sqrt{n}\boldsymbol{\iota}_{1+qk}', n\boldsymbol{\iota}_k']')$ *and* $\mathbf{G} := [\mathbf{G}_1, \ldots, \mathbf{G}_n]'$. *Then,*

$$\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1} = \sum_{t=1}^{n} \begin{bmatrix} n^{-1} & n^{-1}\bar{\mathbf{W}}_t' & n^{-3/2}\mathbf{X}_t' \\ n^{-1}\bar{\mathbf{W}}_t & n^{-1}\bar{\mathbf{W}}_t\bar{\mathbf{W}}_t' & n^{-3/2}\bar{\mathbf{W}}_t\mathbf{X}_t' \\ n^{-3/2}\mathbf{X}_t & n^{-3/2}\mathbf{X}_t\bar{\mathbf{W}}_t & n^{-2}\mathbf{X}_t\mathbf{X}_t' \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & \mathbf{0}' & \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)'dr \\ \mathbf{0} & \mathbb{E}\left[\bar{\mathbf{W}}_t\bar{\mathbf{W}}_t'\right] & \mathbf{0}' \\ \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)dr & \mathbf{0} & \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)\bar{\boldsymbol{\mathcal{B}}}_W(r)'dr \end{bmatrix};$$

(ii) Furthermore,

$$\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U}) = \sum_{t=1}^{n} \begin{bmatrix} n^{-1/2}\psi_\tau[U_t(\tau)] \\ n^{-1/2}\psi_\tau[U_t(\tau)]\bar{\mathbf{W}}_t \\ n^{-1}\psi_\tau[U_t(\tau)]\mathbf{X}_t \end{bmatrix} \Rightarrow \begin{bmatrix} \mathcal{B}_\psi(1,\tau) \\ \boldsymbol{\mathcal{B}}_{\psi\cdot\mathbf{W}}(1,\tau) \\ \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r) d\mathcal{B}_\psi(r,\tau) \end{bmatrix};$$

(iii) $\mathbf{D}_{\mathbf{H}}^{-1}\mathbf{G}'\mathbf{K}(\tau) = O_{\mathbb{P}}(1)$*, where* $\mathbf{D}_{\mathbf{H}} := \mathrm{diag}([n\boldsymbol{\iota}_{1+qk}', n^{3/2}\boldsymbol{\iota}_k']')$*;*

(iv) $\mathbf{M} := n^{-2}\mathbf{X}'[\mathbf{I} - \widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}']\mathbf{X} \Rightarrow \int_0^1 \widetilde{\boldsymbol{\mathcal{B}}}_W(r)\widetilde{\boldsymbol{\mathcal{B}}}_W(r)'dr$*; and*

(v) $n^{-1}\mathbf{X}'[\mathbf{I} - \widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}']\boldsymbol{\Psi}_\tau(\mathbf{U}) \Rightarrow \int_0^1 \widetilde{\boldsymbol{\mathcal{B}}}_W(r)d\mathcal{B}_\psi(r,\tau)$. □

Proof of Corollary A1: (i) Lemmas A1(i), A2(i) and A2(ii) imply that

$$\left\{n^{-1}\sum_{t=1}^{n}\bar{\mathbf{W}}_t, n^{-1}\sum_{t=1}^{n}\bar{\mathbf{W}}_t\bar{\mathbf{W}}_t', n^{-3/2}\sum_{t=1}^{n}\bar{\mathbf{W}}_t\mathbf{X}_t'\right\} \to_{\mathbb{P}} \left\{\mathbf{0}, \mathbb{E}[\bar{\mathbf{W}}_t\bar{\mathbf{W}}_t'], \mathbf{0}\right\}.$$

Next, Lemma A3 implies that

$$\left\{n^{-3/2}\mathbf{X}_t, n^{-2}\mathbf{X}_t'\mathbf{X}_t\right\} \Rightarrow \left\{\int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)dr, \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)\bar{\boldsymbol{\mathcal{B}}}_W(r)dr\right\}.$$

Combining these two results we obtain the desired result in Corollary A1(i).

(ii) Assumption 1(vi) implies that

$$\left\{n^{-1/2}\sum_{t=1}^{n}\psi_\tau[U_t(\tau)], n^{-1/2}\sum_{t=1}^{n}\psi_\tau[U_t(\tau)]\bar{\mathbf{W}}_t\right\} \Rightarrow \left\{\mathcal{B}_\psi(1,\tau), \boldsymbol{\mathcal{B}}_{\psi\cdot\mathbf{W}}(1,\tau)\right\}.$$

Moreover, Lemma A3 implies that $n^{-1}\sum_1^n \psi_\tau[U_t(\tau)]\mathbf{X}_t \Rightarrow \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)d\mathcal{B}_\psi(r,\tau)$. By combining these results, we show that the asymptotic limit of $\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U})$ is equal to Corollary A1(ii).

(iii) Notice that $\mathbf{D}_{\mathbf{H}}^{-1}\mathbf{G}'\mathbf{K}(\tau) = [n^{-1}\sum_{t=1}^n \mathbf{K}_t(\tau)\widetilde{\mathbf{W}}_t', n^{-3/2}\sum_{t=1}^n \mathbf{K}_t(\tau)\mathbf{X}_t']'$. Then, the desired result in Corollary A1(iii) follows from Lemmas A1(ii), A2(iv) and A2(v).

(iv) Note that

$$\mathbf{M} = n^{-2}\mathbf{X}'[\mathbf{I} - \widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}']\mathbf{X} = n^{-2}\mathbf{X}'\mathbf{X} - (n^{-3/2}\mathbf{X}'\widetilde{\mathbf{W}})(n^{-1}\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}(n^{-3/2}\widetilde{\mathbf{W}}'\mathbf{X}).$$

By Lemma A3, we have $n^{-2}\mathbf{X}'\mathbf{X} = O_P(1)$; by Corollary A1(i) and Assumption 1(vi), $(n^{-1}\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1} \to$

diag$[1,\mathbb{E}(\bar{\mathbf{W}}_t\bar{\mathbf{W}}_t')^{-1}]$ *a.s.*; and by Corollary A1(i) $n^{-3/2}\widetilde{\mathbf{W}}'\mathbf{X} \Rightarrow [\int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)dr, \mathbf{0}]$. Therefore, we have

$$\begin{aligned}\mathbf{M} &= n^{-2}\mathbf{X}'\mathbf{X} - (n^{-3/2}\mathbf{X}'\boldsymbol{\iota}_n)(n^{-3/2}\mathbf{X}'\boldsymbol{\iota}_n)' + o_{\mathbb{P}}(1)\\ &\Rightarrow \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)\bar{\boldsymbol{\mathcal{B}}}_W(r)'dr - \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)dr \int \bar{\boldsymbol{\mathcal{B}}}_W(r)'dr = \int_0^1 \widetilde{\boldsymbol{\mathcal{B}}}_W(r)\widetilde{\boldsymbol{\mathcal{B}}}_W(r)'dr.\end{aligned}$$

(v) We note that $n^{-1}\mathbf{X}'\boldsymbol{\Psi}_\tau(\mathbf{U}) \Rightarrow \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)d\mathcal{B}_\psi(r,\tau) = O_{\mathbb{P}}(1)$ by Corollary A1(ii), $n^{-3/2}\widetilde{\mathbf{W}}'\mathbf{X} \Rightarrow [\int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)dr, \mathbf{0}]$ by Corollary A1(i), $(n^{-1}\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1} \to$ diag$[1,\mathbb{E}[\bar{\mathbf{W}}_t\bar{\mathbf{W}}_t']^{-1}]$ *a.s.* by Corollary A1(i) and Assumption 1(vi), and $n^{-1/2}\widetilde{\mathbf{W}}'\boldsymbol{\Psi}_\tau(\mathbf{U}) \Rightarrow \int_0^1 d\boldsymbol{\mathcal{B}}_{\psi\cdot\mathbf{W}}(r,\tau) = O_{\mathbb{P}}(1)$ by Corollary A1(ii). Therefore,

$$\begin{aligned}&n^{-1}\mathbf{X}'\boldsymbol{\Psi}_\tau(\mathbf{U}) - n^{-3/2}\mathbf{X}'\widetilde{\mathbf{W}}(n^{-1}\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}n^{-1/2}\widetilde{\mathbf{W}}'\boldsymbol{\Psi}_\tau(\mathbf{U})\\ &\quad\Rightarrow \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)d\mathcal{B}_\psi(r,\tau) - \int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)dr\int_0^1 d\mathcal{B}_\psi(r,\tau) = \int_0^1 \widetilde{\boldsymbol{\mathcal{B}}}_W(r)d\mathcal{B}_\psi(r,\tau).\end{aligned}$$

This completes the proof. ■

Lemma A4. *Under Assumptions 1 and 2,*

(i)

$$n^{-1/2}\sum_{t=1}^{\lfloor n(\cdot)\rfloor}\left[\psi_{\tau_1}\left[U_t(\tau_1)\right],\ldots,\psi_{\tau_s}\left[U_t(\tau_s)\right]\right]' \Rightarrow \left[\mathcal{B}_\psi\left(\cdot,\tau_1\right),\ldots,\mathcal{B}_\psi\left(\cdot,\tau_s\right)\right]'; \quad \textit{and}$$

(ii) $\mathbf{J}(\boldsymbol{\tau}) \Rightarrow \boldsymbol{\mathcal{J}}_\beta(\boldsymbol{\tau})$*, where*

$$\mathbf{J}(\boldsymbol{\tau}) := \begin{bmatrix} \left\{f_{\tau_1}\left(1-\sum_{j=1}^p \phi_{j*}(\tau_1)\right)\right\}^{-1} n^{-1}\mathbf{X}'\left[\mathbf{I}-\widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}'\right]\boldsymbol{\Psi}_{\tau_1}(\mathbf{U})\\ \left\{f_{\tau_2}\left(1-\sum_{j=1}^p \phi_{j*}(\tau_2)\right)\right\}^{-1} n^{-1}\mathbf{X}'\left[\mathbf{I}-\widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}'\right]\boldsymbol{\Psi}_{\tau_2}(\mathbf{U})\\ \vdots \\ \left\{f_{\tau_s}\left(1-\sum_{j=1}^p \phi_{j*}(\tau_s)\right)\right\}^{-1} n^{-1}\mathbf{X}'\left[\mathbf{I}-\widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}'\right]\boldsymbol{\Psi}_{\tau_s}(\mathbf{U}) \end{bmatrix}. \qquad \square$$

Proof of Lemma A4: (i) The result in Lemma A4(i) is obviously implied by Assumption 2.

(ii) We note from the proof of Corollary A1(v) that, for each $j = 1, 2, \ldots, s$,

$$\frac{1}{n}\mathbf{X}'[\mathbf{I}-\widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}']\boldsymbol{\Psi}_{\tau_j}(\mathbf{U}) = \frac{1}{n}\mathbf{X}'\boldsymbol{\Psi}_{\tau_j}(\mathbf{U}) - \left[\frac{1}{n^{3/2}}\sum_{t=1}^n \mathbf{X}_t'\right]\left[\frac{1}{\sqrt{n}}\sum_{t=1}^n \psi_{\tau_j}[U_t(\tau_j)]\right] + o_{\mathbb{P}}(1),$$

where we use that for each j, $n^{-1/2}\widetilde{\mathbf{W}}'\boldsymbol{\Psi}_{\tau_j}(\mathbf{U}) = O_{\mathbb{P}}(1)$ by Assumption 2. Applying Lemma A4(i) and the continuous mapping theorem, we obtain the desired result in Lemma A4(ii). ■

Although Lemmas A3 and A4, and Corollary A1 state the weak convergency results as if they are independent, their weak limits are jointly achieved under Assumptions 1 and 2. This is mainly because the weak limits in Lemmas A3 and A4, and Corollary A1 are the variations of the weak limits jointly obtained by Assumptions 1 and 2, respectively.

Using Lemmas A1–A4, we now prove the main results: Theorems 1–4.

Proof of Theorem 1: (i) We first note that

$$\varrho_\tau \left\{ Y_t - \mathbf{Z}_t' \widetilde{\boldsymbol{\alpha}}_n(\tau) \right\} = \varrho_\tau \left\{ U_t(\tau) - \mathbf{D}_n^{-1} \widetilde{\boldsymbol{v}}_n(\tau)' \mathbf{Z}_t \right\},$$

where we let $\widetilde{\boldsymbol{v}}_n(\tau) := \mathbf{D}_n\{\widetilde{\boldsymbol{\alpha}}_n(\tau) - \boldsymbol{\alpha}_*(\tau)\}$ and $\mathbf{D}_n := \mathrm{diag}([\sqrt{n}\boldsymbol{\iota}_{1+qk}', n\boldsymbol{\iota}_{k+p}']')$. Thus, minimizing $\sum_{t=1}^n \varrho_\tau\{Y_t - \mathbf{Z}_t'\boldsymbol{\alpha}\}$ with respect to $\boldsymbol{\alpha}$ is equivalent to minimizing

$$Q_{\tau,n}(\boldsymbol{v}) := \sum_{t=1}^n \left[\varrho_\tau \left\{ U_t(\tau) - (\mathbf{D}_n^{-1}\boldsymbol{v})'\mathbf{Z}_t \right\} - \varrho_\tau \left\{ U_t(\tau) \right\} \right]$$

with respect to $\boldsymbol{v}$. Notice that this objective function can be rewritten as

$$\begin{aligned} Q_{\tau,n}(\boldsymbol{v}) = & -\sum_{t=1}^n \boldsymbol{v}'\mathbf{D}_n^{-1}\mathbf{Z}_t \psi_\tau[U_t(\tau)] \\ & + \sum_{t=1}^n \left\{ U_t(\tau) - \boldsymbol{v}'\mathbf{D}_n^{-1}\mathbf{Z}_t \right\} I\left[\boldsymbol{v}'\mathbf{D}_n^{-1}\mathbf{Z}_t < U_t(\tau) < 0 \right] \\ & - \sum_{t=1}^n \left\{ U_t(\tau) - \boldsymbol{v}'\mathbf{D}_n^{-1}\mathbf{Z}_t \right\} I\left[0 < U_t(\tau) < \boldsymbol{v}'\mathbf{D}_n^{-1}\mathbf{Z}_t) \right] \end{aligned} \tag{A.1}$$

where $\varrho_\tau(x-y) - \varrho_\tau(x) = -y\psi_\tau(x) + (x-y)\left[I(0 > x > y) - I(0 < x < y)\right]$ for $x \neq 0$.

We derive the asymptotic behavior of each element in the RHS of (A.1) by combining the techniques in Pesaran and Shin (1998) and Xiao (2009). First, notice that Assumptions 1(i, ii, iii, and vi) imply Assumptions A, B, and C of Xiao (2009). Thus, we can use his results to prove Theorem 1 as follows:

$$\sum_{t=1}^n \left\{ U_t(\tau) - \boldsymbol{v}'\mathbf{D}_n^{-1}\mathbf{Z}_t \right\} I\left[\boldsymbol{v}'\mathbf{D}_n^{-1}\mathbf{Z}_t < U_t(\tau) < 0 \right] = \frac{1}{2} f_\tau \boldsymbol{v}'\mathbf{D}_n^{-1}(\mathbf{Z}'\mathbf{Z})\mathbf{D}_n^{-1}\boldsymbol{v} I\left[\boldsymbol{v}'\mathbf{D}_n^{-1}\mathbf{Z}_t < 0 \right] + o_{\mathbb{P}}(1)$$

$$\sum_{t=1}^n \left\{ \boldsymbol{v}'\mathbf{D}_n^{-1}\mathbf{Z}_t - U_t(\tau) \right\} I\left[0 < U_t(\tau) < \boldsymbol{v}'\mathbf{D}_n^{-1}\mathbf{Z}_t \right] = \frac{1}{2} f_\tau \boldsymbol{v}'\mathbf{D}_n^{-1}(\mathbf{Z}'\mathbf{Z})\mathbf{D}_n^{-1}\boldsymbol{v} I\left[\boldsymbol{v}'\mathbf{D}_n^{-1}\mathbf{Z}_t > 0 \right] + o_{\mathbb{P}}(1),$$

where $\mathbf{Z} := [\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n]'$. Hence, $Q_{\tau,n}(\boldsymbol{v}) = -\boldsymbol{v}'\mathbf{D}_n^{-1}(\sum_{t=1}^n \mathbf{Z}_t\psi_\tau[U_t(\tau)]) + \frac{1}{2} f_\tau \boldsymbol{v}'\mathbf{D}_n^{-1}(\mathbf{Z}'\mathbf{Z})\mathbf{D}_n^{-1}\boldsymbol{v}$ $+o_{\mathbb{P}}(1)$, implying that $\mathbf{D}_n^{-1}\widetilde{\boldsymbol{v}}_n(\tau) = \widetilde{\boldsymbol{\alpha}}_n(\tau) - \boldsymbol{\alpha}_*(\tau) = f_\tau^{-1}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\Psi}_\tau(\mathbf{U}) + o_{\mathbb{P}}(1)$.

Next, partitioning $\mathbf{Z}$ into $[\mathbf{G}, \widetilde{\mathbf{Y}}]$, where $\mathbf{G} := [\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n]'$ and $\widetilde{\mathbf{Y}} := [\widetilde{\mathbf{Y}}_1, \widetilde{\mathbf{Y}}_2, \ldots, \widetilde{\mathbf{Y}}_n]'$, we can show that

$$\widetilde{\boldsymbol{\phi}}_n(\tau) - \boldsymbol{\phi}_*(\tau) = f_\tau^{-1}[\widetilde{\mathbf{Y}}'\mathbf{P}_{\mathbf{G}}\widetilde{\mathbf{Y}}]^{-1}\widetilde{\mathbf{Y}}'\mathbf{P}_{\mathbf{G}}\boldsymbol{\Psi}_\tau(\mathbf{U}),$$

where $\mathbf{P}_{\mathbf{G}} := \mathbf{I} - \mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'$ and $\widetilde{\boldsymbol{\alpha}}_n(\tau) - \boldsymbol{\alpha}_*(\tau) = [(\widetilde{\boldsymbol{\lambda}}_n(\tau) - \boldsymbol{\lambda}_*(\tau))', (\widetilde{\boldsymbol{\phi}}_n(\tau) - \boldsymbol{\phi}_*(\tau))']'$. We also note that $\widetilde{\mathbf{Y}} = \mathbf{G}\boldsymbol{\Gamma}_*(\tau) + \mathbf{K}(\tau)$, where $\mathbf{K}(\tau) := \mathbf{K}(\tau) := [\mathbf{K}_1(\tau), \ldots, \mathbf{K}_n(\tau)]'$. Since $\mathbf{P}_{\mathbf{G}}\mathbf{G} = \mathbf{0}$, hence

$$\widetilde{\mathbf{Y}}'\mathbf{P}_{\mathbf{G}}\widetilde{\mathbf{Y}} = \mathbf{K}(\tau)'\mathbf{K}(\tau) - \mathbf{K}(\tau)'\mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'\mathbf{K}(\tau),$$

$$\widetilde{\mathbf{Y}}'\mathbf{P}_{\mathbf{G}}\boldsymbol{\Psi}_\tau(\mathbf{U}) = \mathbf{K}(\tau)'\boldsymbol{\Psi}_\tau(\mathbf{U}) - \mathbf{K}(\tau)'\mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U}).$$

Furthermore, $n^{-1}\mathbf{K}(\tau)'\mathbf{K}(\tau) \to \mathbb{E}[\mathbf{K}_t(\tau)\mathbf{K}_t(\tau)']$ *a.s.* by Lemma A2(iii), and $\mathbf{K}(\tau)'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1}(\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1})^{-1}\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{K}(\tau) = O_{\mathbb{P}}(n)$ by Corollary A1(iii). More specifically, it follows that $n^{-1/2}\mathbf{K}(\tau)'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1} \Rightarrow [\mathbb{E}[\mathbf{K}_t(\tau)], \mathbb{E}[\mathbf{K}_t(\tau)\bar{\mathbf{W}}_t'], \mathbb{E}[\mathbf{K}_t(\tau)]\int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)'dr]$. Furthermore, note that $n^{1/2}\mathbf{D}_{\mathbf{G}} = \mathbf{D}_{\mathbf{H}}$ and from this, it follows that $\mathbf{K}(\tau)'\mathbf{G}\mathbf{D}_{\mathbf{H}}^{-1}(\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1})^{-1}\mathbf{D}_{\mathbf{H}}^{-1}\mathbf{G}'\mathbf{K}(\tau) \to_{\mathbb{P}} \mathbb{E}[\mathbf{K}_t(\tau)\widetilde{\mathbf{W}}_t']\mathbb{E}[\widetilde{\mathbf{W}}_t\widetilde{\mathbf{W}}_t']^{-1}\mathbb{E}[\widetilde{\mathbf{W}}_t\, \mathbf{K}_t(\tau)']$ by Corollary A1(i). This simple result follows from the fact that the limit matrix of $\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1}$ has zero blocks that prevent the effects of $\int_0^1 \bar{\boldsymbol{\mathcal{B}}}_W(r)'dr$ from being conveyed to the limit. This implies that

$$n^{-1}\widetilde{\mathbf{Y}}'\mathbf{P}_{\mathbf{G}}\widetilde{\mathbf{Y}} \to_{\mathbb{P}} \mathbb{E}\left[\mathbf{K}_t(\tau)\mathbf{K}_t(\tau)'\right] - \mathbb{E}\left[\mathbf{K}_t(\tau)\widetilde{\mathbf{W}}_t'\right]\mathbb{E}\left[\widetilde{\mathbf{W}}_t\widetilde{\mathbf{W}}_t'\right]^{-1}\mathbb{E}\left[\widetilde{\mathbf{W}}_t\mathbf{K}_t(\tau)'\right] = \mathbb{E}\left[\widetilde{\mathbf{H}}_t(\tau)\widetilde{\mathbf{H}}_t(\tau)'\right].$$

In a similar manner, $\mathbf{K}(\tau)'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1}(\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1})^{-1}\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U}) = O_{\mathbb{P}}(\sqrt{n})$ by Corollaries A1(i,ii,iii). Hence,

$$\begin{aligned} n^{-1/2}&\mathbf{K}(\tau)'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1}(\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1})^{-1}\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U}) \\ &= \mathbb{E}\left[\mathbf{K}_t(\tau)\widetilde{\mathbf{W}}_t'\right]\mathbb{E}\left[\widetilde{\mathbf{W}}_t\bar{\mathbf{W}}_t'\right]^{-1}\left(n^{-1/2}\widetilde{\mathbf{W}}'\boldsymbol{\Psi}_\tau(\mathbf{U})\right) + o_{\mathbb{P}}(1), \end{aligned}$$

which implies that

$$\begin{aligned} n^{-1/2}\widetilde{\mathbf{Y}}'\mathbf{P}_{\mathbf{G}}\boldsymbol{\Psi}_\tau(\mathbf{U}) &= n^{-1/2}\mathbf{K}(\tau)'\boldsymbol{\Psi}_\tau(\mathbf{U}) - \mathbb{E}\left[\mathbf{K}_t(\tau)\widetilde{\mathbf{W}}_t'\right]\mathbb{E}\left[\widetilde{\mathbf{W}}_t\widetilde{\mathbf{W}}_t'\right]^{-1} n^{-1/2}\widetilde{\mathbf{W}}'\boldsymbol{\Psi}_\tau(\mathbf{U}) + o_{\mathbb{P}}(1) \\ &= n^{-1/2}\widetilde{\mathbf{H}}(\tau)'\boldsymbol{\Psi}_\tau(\mathbf{U}) + o_{\mathbb{P}}(1) \overset{\text{A}}{\sim} N\left\{\mathbf{0}, \tau(1-\tau)\mathbb{E}\left[\widetilde{\mathbf{H}}_t(\tau)\widetilde{\mathbf{H}}_t(\tau)'\right]\right\} \end{aligned}$$

according to Assumption 1(vi). This in turn implies that

$$\sqrt{n}\left(\widetilde{\boldsymbol{\phi}}_n(\tau)-\boldsymbol{\phi}_*(\tau)\right)\overset{\mathrm{A}}{\sim}N\left[\mathbf{0},\tau(1-\tau)f_\tau^{-2}\mathbb{E}\left[\widetilde{\mathbf{H}}_t(\tau)\widetilde{\mathbf{H}}_t(\tau)'\right]^{-1}\right].$$

We note that the asymptotic variance is identical to $\boldsymbol{\Pi}(\tau)$ by definition.

(ii) To prove Theorem 1(ii), we use the fact that $\widetilde{\mathbf{Y}}=\mathbf{G}\boldsymbol{\Gamma}_*(\tau)+\mathbf{K}(\tau)$. We then show that

$$\begin{aligned}\widetilde{\boldsymbol{\lambda}}_n(\tau)-\boldsymbol{\lambda}_*(\tau)&=f_\tau^{-1}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'\boldsymbol{\Psi}(\mathbf{U})-(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'\widetilde{\mathbf{Y}}\left(\widetilde{\boldsymbol{\phi}}_n(\tau)-\boldsymbol{\phi}_*(\tau)\right)\\&=f_\tau^{-1}\left(\mathbf{G}'\mathbf{G}\right)^{-1}\mathbf{G}'\boldsymbol{\Psi}(\mathbf{U})-\boldsymbol{\Gamma}_*(\tau)\left(\widetilde{\boldsymbol{\phi}}_n(\tau)-\boldsymbol{\phi}_*(\tau)\right)-\left(\mathbf{G}'\mathbf{G}\right)^{-1}\mathbf{G}'\mathbf{K}(\tau)\left(\widetilde{\boldsymbol{\phi}}_n(\tau)-\boldsymbol{\phi}_*(\tau)\right).\end{aligned}\tag{A.2}$$

If we let $\widetilde{\boldsymbol{\varsigma}}_n(\tau):=\widetilde{\boldsymbol{\lambda}}_n(\tau)+\boldsymbol{\Gamma}_*(\tau)\widetilde{\boldsymbol{\phi}}_n(\tau)$ and $\boldsymbol{\varsigma}_*(\tau):=\boldsymbol{\lambda}_*(\tau)+\boldsymbol{\Gamma}_*(\tau)\boldsymbol{\phi}_*(\tau)$, it easily follows that $\widetilde{\boldsymbol{\varsigma}}_n(\tau)-\boldsymbol{\varsigma}_*(\tau):=(\widetilde{\boldsymbol{\lambda}}_n(\tau)-\boldsymbol{\lambda}_*(\tau))+\boldsymbol{\Gamma}_*(\tau)(\widetilde{\boldsymbol{\phi}}_n(\tau)-\boldsymbol{\phi}_*(\tau))$, so we obtain from (A.2) that

$$\begin{aligned}\sqrt{n}\left(\widetilde{\boldsymbol{\varsigma}}_n(\tau)-\boldsymbol{\varsigma}_*(\tau)\right)=&\sqrt{n}f_\tau^{-1}\mathbf{D}_{\mathbf{G}}^{-1}\left(\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1}\right)^{-1}\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U})\\&-\sqrt{n}\mathbf{D}_{\mathbf{G}}^{-1}(\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1})^{-1}\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{K}(\tau)\left(\widetilde{\boldsymbol{\phi}}_n(\tau)-\boldsymbol{\phi}_*(\tau)\right).\end{aligned}$$

Furthermore, Corollaries A1(i, ii) imply that $\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1}=O_{\mathbb{P}}(1)$ and $\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U})=O_{\mathbb{P}}(1)$. Also, $\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{K}(\tau)=O_{\mathbb{P}}(1)$ by Corollary A1(iii). We have already shown that Theorem 1(i) implies that $\widetilde{\boldsymbol{\phi}}_n(\tau)-\boldsymbol{\phi}_*(\tau)=o_{\mathbb{P}}(1)$ and $\sqrt{n}\mathbf{D}_{\mathbf{G}}^{-1}=O(1)$. By combining all of these results, we obtain that

$$\begin{aligned}\sqrt{n}\left(\widetilde{\boldsymbol{\varsigma}}_n(\tau)-\boldsymbol{\varsigma}_*(\tau)\right)=&\sqrt{n}f_\tau^{-1}\mathbf{D}_{\mathbf{G}}^{-1}(\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1})^{-1}\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U})+o_{\mathbb{P}}(1)\\=&f_\tau^{-1}\mathbf{N}\left(\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1}\right)^{-1}\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U})+o_{\mathbb{P}}(1),\end{aligned}\tag{A.3}$$

where $\mathbf{N}:=\mathrm{diag}([\boldsymbol{\iota}_{1+qk}',\mathbf{0}_{k\times1}']')$. Thus, the last k elements of $\sqrt{n}(\widetilde{\boldsymbol{\varsigma}}_n(\tau)-\boldsymbol{\varsigma}_*(\tau))$ are $o_{\mathbb{P}}(1)$, which implies that

$$\sqrt{n}\left\{(\widetilde{\gamma}_n(\tau)-\gamma_*(\tau))+\boldsymbol{\beta}_*(\tau)\boldsymbol{\iota}_p'\left(\widetilde{\boldsymbol{\phi}}_n(\tau)-\boldsymbol{\phi}_*(\tau)\right)\right\}=o_{\mathbb{P}}(1)$$

because

$$
\begin{aligned}
&(\widetilde{\varsigma}_n(\tau) - \varsigma_*(\tau)) \\
&= \begin{bmatrix} \widetilde{\alpha}_n(\tau) - \alpha_*(\tau) \\ \widetilde{\boldsymbol{\delta}}_{0,n}(\tau) - \boldsymbol{\delta}_{0*}(\tau) \\ \vdots \\ \widetilde{\boldsymbol{\delta}}_{q-1,n}(\tau) - \boldsymbol{\delta}_{q-1*}(\tau) \\ \widetilde{\gamma}_n(\tau) - \gamma_*(\tau) \end{bmatrix} + \begin{bmatrix} \mu_*(\tau) & \mu_*(\tau) & \cdots & \mu_*(\tau) \\ \boldsymbol{\xi}_{1,0*}(\tau) & \boldsymbol{\xi}_{2,0*}(\tau) & \cdots & \boldsymbol{\xi}_{p,0*}(\tau) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\xi}_{1,q-1*}(\tau) & \boldsymbol{\xi}_{2,q-1*}(\tau) & \cdots & \boldsymbol{\xi}_{p,q-1*}(\tau) \\ \boldsymbol{\beta}_*(\tau) & \boldsymbol{\beta}_*(\tau) & \cdots & \boldsymbol{\beta}_*(\tau) \end{bmatrix} \left(\widetilde{\boldsymbol{\phi}}_n(\tau) - \boldsymbol{\phi}_*(\tau)\right).
\end{aligned}
$$

By the asymptotic result in Theorem 1(i), the desired result in Theorem 1(ii) follows. ■

Proof of Theorem 2: We focus on the asymptotic behavior of

$$
n\left\{(\widetilde{\gamma}_n(\tau) - \boldsymbol{\gamma}_*(\tau)) + \boldsymbol{\beta}_*(\tau)\sum_{j=1}^{p}\left(\widetilde{\phi}_{j,n}(\tau) - \phi_{j*}(\tau)\right)\right\},
$$

which is equal to the last k elements of $\mathbf{D}_{\mathbf{G}}(\widetilde{\varsigma}_n(\tau) - \varsigma_*(\tau))$ by definition. Notice that

$$
\begin{aligned}
\mathbf{D}_{\mathbf{G}}(\widetilde{\varsigma}_n(\tau) - \varsigma_*(\tau)) &= \left(\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1}\right)^{-1}\left\{f_\tau^{-1}\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U}) - \mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{K}(\tau)\left(\widetilde{\boldsymbol{\phi}}_n(\tau) - \boldsymbol{\phi}_*(\tau)\right)\right\} \\
&= f_\tau^{-1}\left(\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1}\right)^{-1}\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U}) + o_{\mathbb{P}}(1) \qquad \text{(A.4)}
\end{aligned}
$$

because $\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1} = O_{\mathbb{P}}(1)$, $\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{K}(\tau) = O_{\mathbb{P}}(1)$, and $\widetilde{\boldsymbol{\phi}}_n(\tau) - \boldsymbol{\phi}_*(\tau) = O_{\mathbb{P}}(n^{-1/2})$ by Corollaries A1(i), A1(iii), and Theorem 1(i), respectively. The last k elements of $(\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\mathbf{G}\mathbf{D}_{\mathbf{G}}^{-1})^{-1}\mathbf{D}_{\mathbf{G}}^{-1}\mathbf{G}'\boldsymbol{\Psi}_\tau(\mathbf{U})$ are also equal to $\mathbf{M}^{-1}n^{-1}\mathbf{X}'[\mathbf{I} - \widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}']\boldsymbol{\Psi}_\tau(\mathbf{U})$. Therefore, (A.4) implies that

$$
\begin{aligned}
&n\left\{(\widetilde{\gamma}_n(\tau) - \boldsymbol{\gamma}_*(\tau)) + \boldsymbol{\beta}_*(\tau)\sum_{j=1}^{p}\left(\widetilde{\phi}_{j,n}(\tau) - \phi_{j*}(\tau)\right)\right\} \\
&\quad = f_\tau^{-1}\mathbf{M}^{-1}\left\{n^{-1}\mathbf{X}'[\mathbf{I} - \widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}']\boldsymbol{\Psi}_\tau(\mathbf{U})\right\} + o_{\mathbb{P}}(1) = O_{\mathbb{P}}(1). \qquad \text{(A.5)}
\end{aligned}
$$

Second, we focus on the relationship between $\{(\widetilde{\gamma}_n(\tau) - \boldsymbol{\gamma}_*(\tau)) + \boldsymbol{\beta}_*(\tau)\sum_{j=1}^{p}(\widetilde{\phi}_{j,n}(\tau) - \phi_{j*}(\tau))\}$

and $(\widetilde{\boldsymbol{\beta}}_n(\tau) - \boldsymbol{\beta}_*(\tau))$. Using the identity in [10], we have

$$n\left(\widetilde{\boldsymbol{\beta}}_n(\tau) - \boldsymbol{\beta}_*(\tau)\right)$$
$$= \left\{1 - \sum_{j=1}^{p} \phi_{j*}(\tau)\right\}^{-1} n\left\{(\widetilde{\boldsymbol{\gamma}}_n(\tau) - \boldsymbol{\gamma}_*(\tau)) + \boldsymbol{\beta}_*(\tau)\sum_{j=1}^{p}\left(\widetilde{\phi}_{j,n}(\tau) - \phi_{j*}(\tau)\right)\right\} + o_{\mathbb{P}}(1) \quad \text{(A.6)}$$

because Theorem 1(i) implies that $\widetilde{\boldsymbol{\phi}}_n(\tau) = \boldsymbol{\phi}_*(\tau) + o_{\mathbb{P}}(1)$, $|\sum_{j=1}^{q}\phi_{j*}(\tau)| < 1$ by Assumption 1(v), and $n\{(\widetilde{\boldsymbol{\gamma}}_n(\tau) - \boldsymbol{\gamma}_*(\tau)) + \boldsymbol{\beta}_*(\tau)\sum_{j=1}^{p}(\widetilde{\phi}_{j,n}(\tau) - \phi_{j*}(\tau))\} = O_{\mathbb{P}}(1)$ by (A.5). Finally, combining the results in (A.5) and (A.6), we obtain:

$$n\left(\widetilde{\boldsymbol{\beta}}_n(\tau) - \boldsymbol{\beta}_*(\tau)\right)$$
$$= \left\{f_\tau\left(1 - \sum_{j=1}^{p}\phi_{j*}(\tau)\right)\right\}^{-1}\mathbf{M}^{-1}\left\{n^{-1}\mathbf{X}'[\mathbf{I} - \widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}']\boldsymbol{\Psi}_\tau(\mathbf{U})\right\} + o_{\mathbb{P}}(1) \quad \text{(A.7)}$$

the result of which, combined with Corollary A1(iv,v), and the continuous mapping theorem, delivers the desired result in Theorem 2(i).

We turn to proving Theorem 2(ii). By the result in (A.7) and using the fact that $\mathbf{M} = O_{\mathbb{P}}(1)$ by Corollary A1(iv), we obtain that

$$n\mathbf{M}^{1/2}\left(\widetilde{\boldsymbol{\beta}}_n(\tau) - \boldsymbol{\beta}_*(\tau)\right)$$
$$= \left\{f_\tau\left(1 - \sum_{j=1}^{p}\phi_{j*}(\tau)\right)\right\}^{-1}\mathbf{M}^{-1/2}\left\{n^{-1}\mathbf{X}'[\mathbf{I} - \widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}']\boldsymbol{\Psi}_\tau(\mathbf{U})\right\} + o_{\mathbb{P}}(1).$$

Corollary A1(iv,v) provides the asymptotic limits of $\mathbf{M}$ and $n^{-1}\mathbf{X}'[\mathbf{I} - \widetilde{\mathbf{W}}(\widetilde{\mathbf{W}}'\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{W}}']\boldsymbol{\Psi}_\tau(\mathbf{U})$. Noting that $\widetilde{\mathbf{W}}$ and $\boldsymbol{\Psi}_\tau(\mathbf{U})$ are independent, the desired result in Theorem 2(ii) follows using the mixture normality of Phillips (1991b). ■

Proof of Theorem 3: (i) It follows from [7] that for each $j = 1, 2, \ldots, s$,

$$\sqrt{n}\left(\widetilde{\boldsymbol{\phi}}_n(\tau_j) - \boldsymbol{\phi}_*(\tau_j)\right) = f_{\tau_j}^{-1}\mathbb{E}\left[\widetilde{\mathbf{H}}_t(\tau_j)\widetilde{\mathbf{H}}_t(\tau_j)'\right]^{-1}\left(n^{-1/2}\widetilde{\mathbf{H}}(\tau_j)'\boldsymbol{\Psi}_{\tau_j}(\mathbf{U})\right) + o_{\mathbb{P}}(1). \quad \text{(A.8)}$$

Assumption 1(vi) implies that $\mathbb{E}[\widetilde{\mathbf{H}}_t(\tau_j)\widetilde{\mathbf{H}}_t(\tau_j)']$ is positive definite. Using (A.8), we also note that the asymptotic covariance matrix between $\sqrt{n}(\widetilde{\boldsymbol{\phi}}_n(\tau_i) - \boldsymbol{\phi}_*(\tau_i))$ and $\sqrt{n}(\widetilde{\boldsymbol{\phi}}_n(\tau_j) - \boldsymbol{\phi}_*(\tau_j))$ is obtained as

$$f_{\tau_i}^{-1} f_{\tau_j}^{-1} (\min[\tau_i, \tau_j] - \tau_i \tau_j) \mathbb{E}\left[\widetilde{\mathbf{H}}_t(\tau_i)\widetilde{\mathbf{H}}_t(\tau_i)'\right]^{-1} \mathbb{E}\left[\widetilde{\mathbf{H}}_t(\tau_i)\widetilde{\mathbf{H}}_t(\tau_j)\right] \mathbb{E}\left[\widetilde{\mathbf{H}}_t(\tau_j)\widetilde{\mathbf{H}}_t(\tau_j)'\right]^{-1}$$
$$= f_{\tau_i}^{-1} f_{\tau_j}^{-1} (\min[\tau_i, \tau_j] - \tau_i \tau_j) \mathbf{L}(\tau_i, \tau_i)^{-1} \mathbf{L}(\tau_i, \tau_j) \mathbf{L}(\tau_j, \tau_j)^{-1}$$

by the definition of $\mathbf{L}(\tau_i, \tau_j)$. That is, the i-th row and j-th column block matrix of $\boldsymbol{\Xi}(\boldsymbol{\tau})$ are obtained. By Assumption 2(i), $\boldsymbol{\Xi}(\boldsymbol{\tau})$ is positive definite, and we can apply the multivariate CLT using this to obtain that $\sqrt{n}(\widetilde{\boldsymbol{\Phi}}_n(\boldsymbol{\tau}) - \boldsymbol{\Phi}_*(\boldsymbol{\tau})) \overset{\text{A}}{\sim} N[\mathbf{0}, \boldsymbol{\Xi}(\boldsymbol{\tau})]$.

(ii) By [9], for each $j = 1, \ldots, s$, $\sqrt{n}\{(\widetilde{\gamma}_n(\tau_j) - \gamma_*(\tau_j)) + \boldsymbol{\beta}_*(\tau_j)\boldsymbol{\iota}_p'(\widetilde{\boldsymbol{\phi}}_n(\tau_j) - \boldsymbol{\phi}_*(\tau_j))\} = o_{\mathbb{P}}(1)$, which also implies that $\sqrt{n}[\widetilde{\boldsymbol{\Gamma}}_n(\boldsymbol{\tau}) - \boldsymbol{\Gamma}_*(\boldsymbol{\tau})] = -\sqrt{n}[\boldsymbol{\Lambda}(\boldsymbol{\tau})(\boldsymbol{\Phi}_n(\boldsymbol{\tau}) - \boldsymbol{\Phi}_*(\boldsymbol{\tau}))] + o_{\mathbb{P}}(1)$. Hence, Theorem 3(i) implies that $\sqrt{n}[\widetilde{\boldsymbol{\Gamma}}_n(\boldsymbol{\tau}) - \boldsymbol{\Gamma}_*(\boldsymbol{\tau})] \overset{\text{A}}{\sim} N(\mathbf{0}, \boldsymbol{\Lambda}(\boldsymbol{\tau})\boldsymbol{\Xi}(\boldsymbol{\tau})\boldsymbol{\Lambda}(\boldsymbol{\tau})')$. This completes the proof. ■

Proof of Theorem 4: (i) It is easily seen from the definition of $\widetilde{\mathbf{B}}_n(\boldsymbol{\tau})$ and the result in (A.7) that

$$n\left[\widetilde{\mathbf{B}}_n(\boldsymbol{\tau}) - \mathbf{B}_*(\boldsymbol{\tau})\right] = [\mathbf{I}_s \otimes \mathbf{M}^{-1}]\mathbf{J}(\boldsymbol{\tau}) + o_{\mathbb{P}}(1), \tag{A.9}$$

and $\mathbf{J}(\boldsymbol{\tau}) \Rightarrow \boldsymbol{\mathcal{J}}_{\boldsymbol{\beta}}(\boldsymbol{\tau})$ by Lemma A4(ii). Thus, Corollary A1(iv) implies that

$$n\left[\widetilde{\mathbf{B}}_n(\boldsymbol{\tau}) - \mathbf{B}_*(\boldsymbol{\tau})\right] \Rightarrow \left[\mathbf{I}_s \otimes \left(\int_0^1 \widetilde{\boldsymbol{\mathcal{B}}}_W(r)\widetilde{\boldsymbol{\mathcal{B}}}_W(r)'dr\right)^{-1}\right] \boldsymbol{\mathcal{J}}_{\boldsymbol{\beta}}(\boldsymbol{\tau}).$$

(ii) Using (A.9), we note that $\text{rank}[\mathbf{I}_s \otimes \mathbf{M}^{1/2}] = ks$ implies that $n[\mathbf{I}_s \otimes \mathbf{M}^{1/2}][\widetilde{\mathbf{B}}_n(\boldsymbol{\tau}) - \mathbf{B}_*(\boldsymbol{\tau})] = [\mathbf{I}_s \otimes \mathbf{M}^{-1/2}]\mathbf{J}(\boldsymbol{\tau}) + o_{\mathbb{P}}(1)$. Hence, applying the continuous mapping theorem, we obtain:

$$n\left[\mathbf{I}_s \otimes \mathbf{M}^{1/2}\right]\left[\widetilde{\mathbf{B}}_n(\boldsymbol{\tau}) - \mathbf{B}_*(\boldsymbol{\tau})\right] \Rightarrow \left[\mathbf{I}_s \otimes \left(\int_0^1 \widetilde{\boldsymbol{\mathcal{B}}}_W(r)\widetilde{\boldsymbol{\mathcal{B}}}_W(r)'dr\right)^{-1/2}\right] \boldsymbol{\mathcal{J}}_{\boldsymbol{\beta}}(\boldsymbol{\tau}).$$

For each $i = 1, 2, \ldots, s$, $\mathcal{B}_\psi(\cdot, \tau_i)$ is independent of $\widetilde{\boldsymbol{\mathcal{B}}}_W(\cdot)$, so that we can obtain the desired result by applying the mixture interpretation of Phillips (1991b). ■

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