Online Supplement for

'Estimation and Inference of Nonlinear Autoregressive Distributed Lag Models with Time Trend by Ordinary Least Squares'

by

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This Online Supplement is an Appendix that provides proofs of all the results in the paper, including the lemmas, additional Monte Carlo simulations, and supplementary empirical findings.

A Supplements

A.1 Supplementary Lemmas

Before proving the main claims in the text, we first provide some preliminary lemmas to prove the main claims efficiently.

The following two lemmas provide alternative representations of OLS under different environments. They are obtained by elaborating on the Frisch-Waugh-Lovell projection method. We first suppose that $y_t \in \mathbb{R}$ is a dependent variable and $(\boldsymbol{x}_t', \boldsymbol{z}_t')' \in \mathbb{R}^{(s+k)}$ is an explanatory variable, and the OLS estimator is obtained by regressing y_t against $(\boldsymbol{x}_t', \boldsymbol{z}_t')'$. Given this, we provide alternative forms of the OLS estimator in the following lemmas:

Lemma A.1 (Complete Form of Frisch-Waugh-Lovell)**.** *Suppose that* $\{(y_t, \boldsymbol{x}_t', \boldsymbol{z}_t')' \in \mathbb{R}^{1+s+k} : t = 1, 2, \ldots, T\}$. *If the OLS estimators are obtained as follows: for* $j = 1, 2, \ldots, s$*,*

$$(\widetilde{\boldsymbol{\beta}}_T, \widehat{\boldsymbol{\gamma}}_T) := \underset{\boldsymbol{\beta},\boldsymbol{\gamma}}{\arg\min} \sum_{t=1}^{T} \left(y_t - \boldsymbol{x}_t\boldsymbol{\beta} - \boldsymbol{z}_t'\boldsymbol{\gamma}\right)^2, \quad \widehat{\boldsymbol{\phi}}_{jT} := \underset{\boldsymbol{\phi}_j}{\arg\min} \sum_{t=1}^{T} \left(x_{jt} - \boldsymbol{z}_t'\boldsymbol{\phi}_j\right)^2, \quad \textit{and}$$

$$(\widehat{\boldsymbol{\xi}}_T, \widehat{\boldsymbol{\delta}}_T) := \underset{\boldsymbol{\xi},\boldsymbol{\delta}}{\arg\min} \sum_{t=1}^{T} \left(y_t - \widehat{\boldsymbol{v}}_t'\boldsymbol{\xi} - \boldsymbol{z}_t'\boldsymbol{\delta}\right)^2,$$

where for each t*,* $\widehat{\boldsymbol{v}}_t := \boldsymbol{x}_t - \widehat{\boldsymbol{\phi}}_T'\boldsymbol{z}_t$ *and* $\widehat{\boldsymbol{\phi}}_T := (\widehat{\boldsymbol{\phi}}_{1T}, \ldots, \widehat{\boldsymbol{\phi}}_{sT})$*, then* $\widetilde{\boldsymbol{\beta}}_T = \widehat{\boldsymbol{\xi}}_T$ *and* $\widehat{\boldsymbol{\gamma}}_T = \widehat{\boldsymbol{\delta}}_T - \widehat{\boldsymbol{\phi}}_T\widehat{\boldsymbol{\xi}}_T$. $\square$

Therefore, we can obtain the OLS estimator $(\widetilde{\boldsymbol{\beta}}_T, \widehat{\boldsymbol{\gamma}}_T)$ by combining the two OLS estimators $\widehat{\boldsymbol{\phi}}_T$ and $(\widehat{\boldsymbol{\xi}}_T, \widehat{\boldsymbol{\delta}}_T)$ obtained from the first- and second-step OLS estimators, respectively.

The following lemma assumes a different environment. We let the OLS estimator be obtained by regressing y_t against $(\boldsymbol{x}_t, \boldsymbol{z}_t', \boldsymbol{w}_t')' \in \mathbb{R}^{s+k+d}$ and provide another alternative form of the OLS estimator:

Lemma A.2 (Augmented Frisch-Waugh-Lovell, AFWL). *Suppose that* $\{(y_t, \boldsymbol{x}_t', \boldsymbol{z}_t', \boldsymbol{w}_t')' \in \mathbb{R}^{1+s+k+d} : t = 1, 2, \ldots, T\}$. *If the OLS estimators are obtained as follows: for* $j = 1, 2, \ldots, s$,

$$(\widetilde{\boldsymbol{\beta}}_T, \widehat{\boldsymbol{\gamma}}_T, \widehat{\boldsymbol{\alpha}}_T) := \arg\min_{\beta, \boldsymbol{\gamma}, \boldsymbol{\alpha}} \sum_{t=1}^{T} \left(y_t - \boldsymbol{x}_t'\boldsymbol{\beta} - \boldsymbol{z}_t'\boldsymbol{\gamma} - \boldsymbol{w}_t'\boldsymbol{\alpha}\right)^2, \quad \widehat{\boldsymbol{\phi}}_{jT} := \arg\min_{\boldsymbol{\phi}} \sum_{t=1}^{T} \left(x_{jt} - \boldsymbol{z}_t'\boldsymbol{\phi}_j\right)^2, \quad \textit{and}$$

$$(\widehat{\boldsymbol{\xi}}_T, \widehat{\boldsymbol{\delta}}_T, \widehat{\boldsymbol{\theta}}_T) := \arg\min_{\boldsymbol{\xi}, \boldsymbol{\delta}, \boldsymbol{\theta}} \sum_{t=1}^{T} \left(y_t - \widehat{\boldsymbol{v}}_t'\boldsymbol{\xi} - \boldsymbol{z}_t'\boldsymbol{\delta} - \boldsymbol{w}_t'\boldsymbol{\theta}\right)^2,$$

where for each t, $\widehat{\boldsymbol{v}}_t := \boldsymbol{x}_t - \widehat{\boldsymbol{\phi}}_T'\boldsymbol{z}_t$ *and* $\widehat{\boldsymbol{\phi}}_T := (\widehat{\boldsymbol{\phi}}_{1T}, \ldots, \widehat{\boldsymbol{\phi}}_{sT})$, *then* $\widetilde{\boldsymbol{\beta}}_T = \widehat{\boldsymbol{\xi}}_T$, $\widehat{\boldsymbol{\gamma}}_T = \widehat{\boldsymbol{\delta}}_T - \widehat{\boldsymbol{\phi}}_T\widehat{\boldsymbol{\xi}}_T$, *and* $\widehat{\boldsymbol{\alpha}}_T = \widehat{\boldsymbol{\theta}}_T$. □

Note that $\boldsymbol{w}_t$ is added as an additional regressor to the regressors given in Lemma A.1 and that the nuisance parameter estimator $\widehat{\boldsymbol{\alpha}}_T$ is the same as the nuisance parameter estimator $\widehat{\boldsymbol{\theta}}_T$ obtained in the second step.

We now prove Lemmas A.1 and A.2. For notational simplicity, we let

$$\mathbf{Y} := [y_1, y_2, \ldots, y_T]', \quad \mathbf{X} := [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_T]', \quad \mathbf{Z} := [\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_T]', \quad \widehat{\mathbf{V}} := [\widehat{v}_1, \widehat{v}_2, \ldots, \widehat{v}_T]',$$

and $\mathbf{W} := [\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_T]'$.

Proof of Lemma A.1. From the definition of $(\widetilde{\boldsymbol{\beta}}_T, \widehat{\boldsymbol{\gamma}}_T)$, we first note that

$$\begin{bmatrix} \widetilde{\boldsymbol{\beta}}_T \\ \widehat{\boldsymbol{\gamma}}_T \end{bmatrix} = \begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z} \\ \mathbf{Z}'\mathbf{X} & \mathbf{Z}'\mathbf{Z} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X}'\mathbf{Y} \\ \mathbf{Z}'\mathbf{Y} \end{bmatrix} = \begin{bmatrix} (\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}\mathbf{X}'\mathbf{Q}\mathbf{Y} \\ (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}\mathbf{X}'\mathbf{Q}]\mathbf{Y} \end{bmatrix}, \tag{A.1}$$

where $\mathbf{Q} := \mathbf{I} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'$. Next, we note that

$$\widehat{\mathbf{V}} = \mathbf{X} - \mathbf{Z}\widehat{\boldsymbol{\phi}}_T = \mathbf{X} - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X} = \mathbf{Q}\mathbf{X}. \tag{A.2}$$

Therefore, $\mathbf{Z}'\widehat{\mathbf{V}} = \mathbf{0}$ by noting that $\mathbf{Z}'\mathbf{Q} = \mathbf{0}$. Third, we note that

$$\begin{bmatrix} \widehat{\boldsymbol{\xi}}_T \\ \widehat{\boldsymbol{\delta}}_T \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{V}}'\widehat{\mathbf{V}} & \widehat{\mathbf{V}}'\mathbf{Z} \\ \mathbf{Z}'\widehat{\mathbf{V}} & \mathbf{Z}'\mathbf{Z} \end{bmatrix}^{-1} \begin{bmatrix} \widehat{\mathbf{V}}'\mathbf{Y} \\ \mathbf{Z}'\mathbf{Y} \end{bmatrix} = \begin{bmatrix} (\widehat{\mathbf{V}}'\widehat{\mathbf{V}})^{-1}\widehat{\mathbf{V}}'\mathbf{Y} \\ (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y} \end{bmatrix}$$

using the fact that $\mathbf{Z}'\widehat{\mathbf{V}} = \mathbf{0}$. Therefore,

$$\widehat{\boldsymbol{\xi}}_T = (\widehat{\mathbf{V}}'\widehat{\mathbf{V}})^{-1}\widehat{\mathbf{V}}'\mathbf{Y} = (\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}\mathbf{X}'\mathbf{Q}\mathbf{Y} \tag{A.3}$$

using (A.2). This shows that $\widetilde{\boldsymbol{\beta}}_T = \widehat{\boldsymbol{\xi}}_T$. Finally, we note that

$$\widehat{\boldsymbol{\delta}}_T - \widehat{\boldsymbol{\phi}}_T\widehat{\boldsymbol{\xi}}_T = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y} - (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}(\widehat{\mathbf{V}}'\widehat{\mathbf{V}})^{-1}\widehat{\mathbf{V}}'\mathbf{Y} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{Q}\mathbf{X})^{-1}\mathbf{X}'\mathbf{Q}]\mathbf{Y} = \widehat{\boldsymbol{\gamma}}_T,$$

which follows from (A.1), where the second equality follows from (A.3). Thus, $\widehat{\boldsymbol{\delta}}_T - \widehat{\boldsymbol{\phi}}_T\widehat{\boldsymbol{\xi}}_T = \widehat{\boldsymbol{\gamma}}_T$. This completes the proof. ■

Proof of Lemma A.2. To prove the claim, we represent the OLS estimators in different forms. If we let

$$(\widetilde{\boldsymbol{\beta}}_T(\boldsymbol{\alpha}), \widehat{\boldsymbol{\gamma}}_T(\boldsymbol{\alpha})) := \arg\min_{\boldsymbol{\beta},\boldsymbol{\gamma}} \sum_{t=1}^{T} \left(y_t - \boldsymbol{x}_t'\boldsymbol{\beta} - \boldsymbol{z}_t'\boldsymbol{\gamma} - \boldsymbol{w}_t'\boldsymbol{\alpha}\right)^2,$$

then

$$\widehat{\boldsymbol{\alpha}}_T = \arg\min_{\boldsymbol{\alpha}} \sum_{t=1}^{T} \left(y_t - \boldsymbol{x}_t\widetilde{\boldsymbol{\beta}}_T(\boldsymbol{\alpha}) - \boldsymbol{z}_t'\widehat{\boldsymbol{\gamma}}_T(\boldsymbol{\alpha}) - \boldsymbol{w}_t'\boldsymbol{\alpha}\right)^2,$$

and $(\widetilde{\boldsymbol{\beta}}_T(\widehat{\boldsymbol{\alpha}}_T), \widehat{\boldsymbol{\gamma}}_T(\widehat{\boldsymbol{\alpha}}_T)) = (\widetilde{\boldsymbol{\beta}}_T, \widehat{\boldsymbol{\gamma}}_T)$. Likewise, if we let

$$(\widehat{\boldsymbol{\xi}}_T(\boldsymbol{\theta}), \widehat{\boldsymbol{\delta}}_T(\boldsymbol{\theta})) := \arg\min_{\xi,\boldsymbol{\delta}} \sum_{t=1}^{T} \left(y_t - \widehat{\boldsymbol{v}}_t'\boldsymbol{\xi} - \boldsymbol{z}_t'\boldsymbol{\delta} - \boldsymbol{w}_t'\boldsymbol{\theta}\right)^2,$$

then

$$\widehat{\boldsymbol{\theta}}_T = \arg\min_{\boldsymbol{\theta}} \sum_{t=1}^{T} \left(y_t - \widehat{\boldsymbol{v}}_t'\widehat{\boldsymbol{\xi}}_T(\boldsymbol{\theta}) - \boldsymbol{z}_t'\widehat{\boldsymbol{\delta}}_T(\boldsymbol{\theta}) - \boldsymbol{w}_t'\boldsymbol{\theta}\right)^2,$$

and $(\widehat{\boldsymbol{\xi}}_T(\widehat{\boldsymbol{\theta}}_T), \widehat{\boldsymbol{\delta}}_T(\widehat{\boldsymbol{\theta}}_T)) = (\widehat{\boldsymbol{\xi}}_T, \widehat{\boldsymbol{\delta}}_T)$.

Here, for each $\boldsymbol{\alpha}$, if we let $y_t(\boldsymbol{\alpha}) := y_t - \boldsymbol{w}_t'\boldsymbol{\alpha}$,

$$(\widetilde{\boldsymbol{\beta}}_T(\cdot), \widehat{\boldsymbol{\gamma}}_T(\cdot)) := \arg\min_{\boldsymbol{\beta},\boldsymbol{\gamma}} \sum_{t=1}^{T} \left(y_t(\cdot) - \boldsymbol{x}_t'\boldsymbol{\beta} - \boldsymbol{z}_t'\boldsymbol{\gamma}\right)^2, \quad \text{and}$$

$$(\widehat{\boldsymbol{\xi}}_T(\cdot), \widehat{\boldsymbol{\delta}}_T(\cdot)) := \arg\min_{\boldsymbol{\xi},\boldsymbol{\delta}} \sum_{t=1}^{T} \left(y_t(\cdot) - \widehat{\boldsymbol{v}}_t'\boldsymbol{\xi} - \boldsymbol{z}_t'\boldsymbol{\delta}\right)^2,$$

so that Lemma A.1 implies that $\widetilde{\boldsymbol{\beta}}_T(\cdot) = \widehat{\boldsymbol{\xi}}_T(\cdot)$ and $\widehat{\boldsymbol{\gamma}}_T(\cdot) = \widehat{\boldsymbol{\delta}}_T(\cdot) - \widehat{\boldsymbol{\phi}}_T \widehat{\boldsymbol{\xi}}_T(\cdot)$. Therefore,

$$\sum_{t=1}^{T}\left(y_t(\cdot) - \widehat{\boldsymbol{v}}_t'\widehat{\boldsymbol{\xi}}_T(\cdot) - \boldsymbol{z}_t'\widehat{\boldsymbol{\delta}}_T(\cdot)\right)^2 = \sum_{t=1}^{T}\left(y_t(\cdot) - (\boldsymbol{x}_t - \widehat{\boldsymbol{\phi}}_T'\boldsymbol{z}_t)'\widehat{\boldsymbol{\xi}}_T(\cdot) - \boldsymbol{z}_t'\widehat{\boldsymbol{\delta}}_T(\cdot)\right)^2$$
$$= \sum_{t=1}^{T}\left(y_t(\cdot) - \boldsymbol{x}_t'\widehat{\boldsymbol{\xi}}_T(\cdot) - \boldsymbol{z}_t'(\widehat{\boldsymbol{\delta}}_T(\cdot) - \widehat{\boldsymbol{\phi}}_T\widehat{\boldsymbol{\xi}}_T(\cdot))\right)^2 = \sum_{t=1}^{T}\left(y_t(\cdot) - \boldsymbol{x}_t'\widetilde{\boldsymbol{\beta}}_T(\cdot) - \boldsymbol{z}_t'\widehat{\boldsymbol{\gamma}}_T(\cdot)\right)^2,$$

implying that

$$\arg\min_{\boldsymbol{\alpha}} \sum_{t=1}^{T}\left(y_t(\boldsymbol{\alpha}) - \widehat{\boldsymbol{v}}_t'\widehat{\boldsymbol{\xi}}_T(\boldsymbol{\alpha}) - \boldsymbol{z}_t'\widehat{\boldsymbol{\delta}}_T(\boldsymbol{\alpha})\right)^2 = \arg\min_{\boldsymbol{\theta}} \sum_{t=1}^{T}\left(y_t(\boldsymbol{\theta}) - \boldsymbol{x}_t'\widetilde{\boldsymbol{\beta}}_T(\boldsymbol{\theta}) - \boldsymbol{z}_t'\widehat{\boldsymbol{\gamma}}_T(\boldsymbol{\theta})\right)^2,$$

viz., $\widehat{\boldsymbol{\alpha}}_T = \widehat{\boldsymbol{\theta}}_T$. Thus, it follows that $\widetilde{\boldsymbol{\beta}}_T(\widehat{\boldsymbol{\alpha}}_T) = \widehat{\boldsymbol{\xi}}_T(\widehat{\boldsymbol{\theta}}_T)$ and $\widehat{\boldsymbol{\gamma}}_T(\widehat{\boldsymbol{\alpha}}_T) = \widehat{\boldsymbol{\delta}}_T(\widehat{\boldsymbol{\theta}}_T) - \widehat{\boldsymbol{\phi}}_T\widehat{\boldsymbol{\xi}}_T(\widehat{\boldsymbol{\theta}}_T)$. That is, $\widetilde{\boldsymbol{\beta}}_T = \widehat{\boldsymbol{\xi}}_T$ and $\widehat{\boldsymbol{\gamma}}_T = \widehat{\boldsymbol{\delta}}_T - \widehat{\boldsymbol{\phi}}_T\widehat{\boldsymbol{\xi}}_T$. This completes the proof. ■

The following lemma shows that $\widetilde{\boldsymbol{v}}_T$ and $\widetilde{\boldsymbol{\omega}}_T$ defined in Section 3 suffer from an asymptotically singular matrix problem.

Lemma A.3. *Given Assumption 1,*

(i) $T^{-1}\sum_{t=1}^{T}\widetilde{u}_{t-1}^2 \xrightarrow{\mathbb{P}} \sigma_u^2 := \mathbb{E}[u_t^2]$;

(ii) $T^{-1/2}\left(\sum_{t=1}^{T}\widetilde{u}_{t-1}\boldsymbol{r}_{t-1}\right)\widetilde{\mathbf{D}}_1^{-1} \Rightarrow \widetilde{\mathbf{M}}_{1u} := \mathbf{0}_{1\times(2k+2)}$, *where* $\widetilde{\mathbf{D}}_1 := diag[T^{3/2}\mathbf{I}_{2k}, T^{3/2}, T^{1/2}]$;

(iii)

$$\widetilde{\mathbf{D}}_1^{-1}\left(\sum_{t=1}^{T}\boldsymbol{r}_{t-1}\boldsymbol{r}_{t-1}'\right)\widetilde{\mathbf{D}}_1^{-1} \Rightarrow \widetilde{\mathscr{M}}_{11} := \begin{bmatrix} \frac{1}{3}\boldsymbol{\mu}_*\boldsymbol{\mu}_*' & \frac{1}{3}\boldsymbol{\mu}_* & \frac{1}{2}\boldsymbol{\mu}_* \\ \frac{1}{3}\boldsymbol{\mu}_*' & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2}\boldsymbol{\mu}_*' & \frac{1}{2} & 1 \end{bmatrix},$$

which is singular;

(iv) $T^{-1/2}\left(\sum_{t=1}^{T}\widetilde{u}_{t-1}\boldsymbol{z}_{2t}\right)\widetilde{\mathbf{D}}_2^{-1} \xrightarrow{\mathbb{P}} \widetilde{\mathbf{M}}_{2u} := \mathbb{E}[u_{t-1}\boldsymbol{z}_{2t}]$, *where* $\widetilde{\mathbf{D}}_2 := diag[T^{1/2}\mathbf{I}_{p+2kq-1}]$;

(v)

$$\widetilde{\mathbf{D}}_2^{-1}\left(\sum_{t=1}^{T}\boldsymbol{z}_{2t}\boldsymbol{r}_{t-1}'\right)\widetilde{\mathbf{D}}_1^{-1} \Rightarrow \widetilde{\mathscr{M}}_{21} := \begin{bmatrix} \frac{1}{2}\delta_*\boldsymbol{\iota}_{p-1}\boldsymbol{\mu}_*' & \frac{1}{2}\delta_*\boldsymbol{\iota}_{p-1} & \delta_*\boldsymbol{\iota}_{p-1} \\ \frac{1}{2}\boldsymbol{\iota}_q\otimes\boldsymbol{\mu}_*\boldsymbol{\mu}_*' & \frac{1}{2}\boldsymbol{\iota}_q\otimes\boldsymbol{\mu}_* & \boldsymbol{\iota}_q\otimes\boldsymbol{\mu}_* \end{bmatrix};$$

(vi) $\widetilde{\mathbf{D}}_2^{-1}\left(\sum_{t=1}^{T}\boldsymbol{z}_{2t}\boldsymbol{z}_{2t}'\right)\widetilde{\mathbf{D}}_2^{-1} \xrightarrow{\mathbb{P}} \widetilde{\mathbf{M}}_{22} := \mathbf{M}_{22}$; *and*

(vii) if we let $\widetilde{\mathbf{D}} := diag[T^{1/2}, \widetilde{\mathbf{D}}_1, \widetilde{\mathbf{D}}_2]$, $\widetilde{\mathbf{D}}^{-1}\left(\sum_{t=1}^{T}\widetilde{\boldsymbol{z}}_t\widetilde{\boldsymbol{z}}_t'\right)\widetilde{\mathbf{D}}^{-1} \Rightarrow \widetilde{\mathscr{M}}$, *where*

$$\widetilde{\mathbf{D}}^{-1}\left(\sum_{t=1}^{T}\widetilde{\boldsymbol{z}}_t\widetilde{\boldsymbol{z}}_t'\right)\widetilde{\mathbf{D}}^{-1} \Rightarrow \widetilde{\mathscr{M}} := \begin{bmatrix} \sigma_u^2 & \widetilde{\mathbf{M}}_{u1} & \widetilde{\mathbf{M}}_{u2} \\ \widetilde{\mathbf{M}}_{1u} & \widetilde{\mathscr{M}}_{11} & \widetilde{\mathscr{M}}_{12} \\ \widetilde{\mathbf{M}}_{2u} & \widetilde{\mathscr{M}}_{21} & \widetilde{\mathbf{M}}_{22} \end{bmatrix},$$

which is singular, where $\widetilde{\mathbf{D}} := diag[T^{1/2}, \widetilde{\mathbf{D}}_1, \widetilde{\mathbf{D}}_2]$, $\widetilde{\mathscr{M}}_{12} := \widetilde{\mathscr{M}}_{21}'$, $\widetilde{\mathbf{M}}_{u1} := \widetilde{\mathbf{M}}_{1u}'$, *and* $\widetilde{\mathbf{M}}_{u2} := \widetilde{\mathbf{M}}_{2u}'$. □

Both $\widetilde{\boldsymbol{\upsilon}}_T$ and $\widetilde{\boldsymbol{\omega}}_T$ suffer from the asymptotically singular matrix problem by Lemma A.3 (*iii* and *vii*). Specifically, every column from the second to $(1+k)$-th columns of $\widetilde{\mathscr{M}}$ is proportional to the $(2+2k)$-th column. Likewise, every column from the first to k-th columns in $\widetilde{\mathscr{M}}_{11}$ is proportional to the $(1+2k)$-th column of $\widetilde{\mathscr{M}}_{11}$.

We now prove Lemma A.3.

Proof of Lemma A.3. (*i*) This follows from Lemma B.6 (*iv*) given in Section A.2.

(*ii*) This follows from Lemmas B.3 (*v*), B.4 (*v*), and B.5 (*iv*) given in Section A.2.

(*iii*) This follows from Lemmas B.3 (*i*, *ii*), B.4 (*i*, *ii*), and B.5 (*i*) given in Section A.2.

(*iv*) This follows from Lemma B.2 (*vii*) given in Section A.2.

(*v*) This follows from Lemma B.2 (*ii*, *iii*, and *iv*) given in Section A.2.

(*vi*) This follows from Lemma B.2 (*i*) given in Section A.2.

(*vii*) The given weak convergence follows from Lemma A.3 (*i*, *ii*, *iii*, *iv*, *v*, and *vi*) given in Section A.2, and the singularity follows from the structure of $\widetilde{\mathscr{M}}$. ■

A.2 Preliminary Lemmas

We next provide preliminary lemmas to prove the main claims efficiently.

Lemma B.1. *Given Assumption 1,* $\mathbf{B}_T(\cdot) := T^{-1/2}\sum_{t=1}^{\lfloor(\cdot)T\rfloor} \boldsymbol{w}_t \Rightarrow \mathscr{B}(\cdot)$. □

Lemma B.2. *Given Assumption 1,*

(i) $T^{-1}\sum_{t=1}^{T} \boldsymbol{z}_{2t}\boldsymbol{z}_{2t}' \stackrel{\mathbb{P}}{\to} \mathbb{E}[\boldsymbol{z}_{2t}\boldsymbol{z}_{2t}']$;

(ii) $T^{-1}\sum_{t=1}^{T} \boldsymbol{z}_{2t} \stackrel{\mathbb{P}}{\to} [\delta_*\boldsymbol{\iota}_{p-1}', \boldsymbol{\iota}_q' \otimes \boldsymbol{\mu}_*']'$;

(iii) $T^{-2}\sum_{t=1}^{T} (t-1)\boldsymbol{z}_{2t} \stackrel{\mathbb{P}}{\to} [\frac{1}{2}\delta_*\boldsymbol{\iota}_{p-1}', \frac{1}{2}\boldsymbol{\iota}_q' \otimes \boldsymbol{\mu}_*']'$;

(iv) $T^{-3}\sum_{t=1}^{T} \ddot{\boldsymbol{x}}_{t-1}\boldsymbol{z}_{2t}' \stackrel{\mathbb{P}}{\to} [\frac{1}{2}\delta_*\boldsymbol{\mu}_*\boldsymbol{\iota}_{p-1}', \frac{1}{2}\boldsymbol{\mu}_*\boldsymbol{\iota}_q' \otimes \boldsymbol{\mu}_*']$;

(v) $T^{-3/2}\sum_{t=1}^{T} \ddot{\mathbf{m}}_{t-1}\boldsymbol{z}_{2t}' \Rightarrow [\delta_* \int \widetilde{\mathscr{B}}_s\boldsymbol{\iota}_{p-1}', \int \widetilde{\mathscr{B}}_s\boldsymbol{\iota}_q' \otimes \boldsymbol{\mu}_*']$;

(vi) $T^{-2}\sum_{t=1}^{T} y_{t-1}\boldsymbol{z}_{2t}' \stackrel{\mathbb{P}}{\to} [\frac{1}{2}\delta_*^2\boldsymbol{\iota}_{p-1}', \frac{1}{2}\delta_*\boldsymbol{\iota}_q' \otimes \boldsymbol{\mu}_*']'$;

(vii) $T^{-1}\sum_{t=1}^{T} \ddot{u}_{t-1}\boldsymbol{z}_{2t} \stackrel{\mathbb{P}}{\to} \mathbb{E}[u_{t-1}\boldsymbol{z}_{2t}]$. □

Lemma B.3. *Given Assumption 1,*

(i) $T^{-2}\sum_{t=1}^{T} t \to \frac{1}{2}$;

(ii) $T^{-2}\sum_{t=1}^{T} \ddot{\boldsymbol{x}}_t \stackrel{\mathbb{P}}{\to} \frac{1}{2}\boldsymbol{\mu}_*$;

(iii) $T^{-3/2}\sum_{t=1}^{T} \ddot{\mathbf{m}}_t \Rightarrow \int \widetilde{\mathscr{B}}_s$;

(iv) $T^{-2}\sum_{t=1}^{T} y_t \stackrel{\mathbb{P}}{\to} \frac{1}{2}\delta_*$;

(v) $\sum_{t=1}^{T} \ddot{u}_{t-1} \equiv 0$. $\square$

Lemma B.4. *Given Assumption 1,*

(i) $T^{-3}\sum_{t=1}^{T} t^2 \to \frac{1}{3}$;

(ii) $T^{-3}\sum_{t=1}^{T} t\ddot{\boldsymbol{x}}_t \xrightarrow{\mathbb{P}} \frac{1}{3}\boldsymbol{\mu}_*$;

(iii) $\sum_{t=1}^{T}(t-1)\ddot{\mathbf{m}}_{t-1} \equiv \mathbf{0}$;

(iv) $T^{-3}\sum_{t=1}^{T} ty_t \xrightarrow{\mathbb{P}} \frac{1}{3}\delta_*$;

(v) $\sum_{t=1}^{T}(t-1)\ddot{u}_{t-1} \equiv 0$. $\square$

Lemma B.5. *Given Assumption 1,*

(i) $T^{-3}\sum_{t=1}^{T} \ddot{\boldsymbol{x}}_t\ddot{\boldsymbol{x}}_t' \xrightarrow{\mathbb{P}} \frac{1}{3}\boldsymbol{\mu}_*\boldsymbol{\mu}_*'$;

(ii) $T^{-2}\sum_{t=1}^{T} \ddot{\boldsymbol{x}}_{t-1}\ddot{\mathbf{m}}_{t-1}' \Rightarrow \int \boldsymbol{\mathscr{B}}_s\bar{\boldsymbol{\mathscr{B}}}_s'$;

(iii) $T^{-3}\sum_{t=1}^{T} \ddot{\boldsymbol{x}}_t y_t \xrightarrow{\mathbb{P}} \frac{1}{3}\delta_*\boldsymbol{\mu}_*$;

(iv) $\sum_{t=1}^{T} \ddot{\boldsymbol{x}}_{t-1}\ddot{u}_{t-1} \equiv \mathbf{0}$. $\square$

Remark. $\int \boldsymbol{\mathscr{B}}_s\bar{\boldsymbol{\mathscr{B}}}_s' = \int \bar{\boldsymbol{\mathscr{B}}}_s\bar{\boldsymbol{\mathscr{B}}}_s' = \int \boldsymbol{\mathscr{B}}_s\boldsymbol{\mathscr{B}}_s' - 3\int r\boldsymbol{\mathscr{B}}_s \int r\boldsymbol{\mathscr{B}}_s'$ from the definition of $\bar{\boldsymbol{\mathscr{B}}}_s(v) := \boldsymbol{\mathscr{B}}_s(v) - 3v\int r\boldsymbol{\mathscr{B}}_s$. $\square$

Lemma B.6. *Given Assumption 1,*

(i) $T^{-2}\sum_{t=1}^{T} \ddot{\mathbf{m}}_{t-1}\ddot{\mathbf{m}}_{t-1}' \Rightarrow \int \boldsymbol{\mathscr{B}}_s\bar{\boldsymbol{\mathscr{B}}}_s'$;

(ii) $\sum_{t=1}^{T} \ddot{\mathbf{m}}_{t-1}\widetilde{u}_{t-1} \equiv \mathbf{0}$;

(iii) $T^{-3}\sum_{t=1}^{T} y_t^2 \xrightarrow{\mathbb{P}} \frac{1}{3}\delta_*^2$;

(iv) $\ddot{\sigma}_{u,T}^2 := T^{-1}\sum_{t=1}^{T} \widetilde{u}_t^2 \xrightarrow{\mathbb{P}} \sigma_u^2 := \mathbb{E}[u_t^2]$. $\square$

Lemma B.7. *Let $\boldsymbol{\varrho}_{m*} := \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\sum_{\tau=1}^{t-1}\mathbb{E}[\boldsymbol{s}_\tau u_t]$. Given Assumption 1,*

(i) $\sqrt{T}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*) \Rightarrow 3\int r\boldsymbol{\mathscr{B}}_s$;

(ii) *for every t,* $\boldsymbol{r}_t'\widetilde{\boldsymbol{v}}_T = \ddot{\boldsymbol{r}}_t'\ddot{\boldsymbol{v}}_T$;

(iii) $T^{-1}\sum_{t=1}^{T} \mathbf{m}_{t-1}u_t \Rightarrow \int \boldsymbol{\mathscr{B}}_s d\mathscr{B}_u + \boldsymbol{\varrho}_{m*}$;

(iv) $T^{-1}\sum_{t=1}^{T} \mathbf{m}_{t-1}e_t \Rightarrow \int \boldsymbol{\mathscr{B}}_s d\mathscr{B}_e$;

(v) $\widehat{\sigma}_{e,T}^2 \xrightarrow{\mathbb{P}} \sigma_e^2 := \mathbb{E}[e_t^2]$. $\square$

We now prove the preliminary Lemmas B.1 to B.7.

Proof of Lemma B.1. This trivially follows theorem 7.30 of White (2001). ∎

Proof of Lemma B.2. (*i*) It holds by the ergodic theorem.

(*ii*) It holds by the ergodic theorem and the fact that $\mathbb{E}[\boldsymbol{z}_{2t}] = [\delta_*\boldsymbol{\iota}_{p-1}', \boldsymbol{\iota}_q' \otimes \boldsymbol{\mu}_*']'$ because $\delta_*\boldsymbol{\iota}_{p-1} = \mathbb{E}[\Delta\boldsymbol{y}_{t-}]$, $\boldsymbol{\iota}_q' \otimes \boldsymbol{\mu}_*^{+\prime} = \mathbb{E}[(\Delta\boldsymbol{x}_t^{+\prime}, \ldots, \Delta\boldsymbol{x}_{t-q+1}^{+\prime})]$, and $\boldsymbol{\iota}_q' \otimes \boldsymbol{\mu}_*^{-\prime} = \mathbb{E}[(\Delta\boldsymbol{x}_t^{-\prime}, \ldots, \Delta\boldsymbol{x}_{t-q+1}^{-\prime})]$.

(*iii*) We note that $T^{-2}\sum_{t=1}^{T}(t-1)\boldsymbol{z}_{2t} = T^{-1}\sum_{t=1}^{T}((t-1)/T)\boldsymbol{z}_{2t}$, and we let $\boldsymbol{q}_t := ((t-1)/T)\boldsymbol{z}_{2t}$ for notational simplicity, which is a heterogeneous process. Therefore, $T^{-1}\sum_{t=1}^{T}(\boldsymbol{q}_t - \mathbb{E}[\boldsymbol{q}_t]) \xrightarrow{\mathbb{P}} \mathbf{0}$. In addition, $T^{-1}\sum_{t=1}^{T}\mathbb{E}[\boldsymbol{q}_t] = T^{-2}\sum_{t=1}^{T}(t-1)\mathbb{E}[\boldsymbol{z}_{2t}] \rightarrow \frac{1}{2}\mathbb{E}[\boldsymbol{z}_{2t}]$, implying that $T^{-1}\sum_{t=1}^{T}\boldsymbol{q}_t \xrightarrow{\mathbb{P}} \frac{1}{2}\mathbb{E}[\boldsymbol{z}_{2t}]$ by White (2001, theorem 3.47), given the DGP condition in Assumption 1. We further note that $\mathbb{E}[\boldsymbol{z}_{2t}] = [\delta_*\boldsymbol{\iota}_{p-1}', \boldsymbol{\iota}_q'\otimes\boldsymbol{\mu}_*']'$, leading to the desired result.

(*iv*) We first note that $\ddot{\boldsymbol{x}}_{t-1} = \boldsymbol{\mu}_*(t-1) + \mathbf{m}_{t-1}$. Therefore, $\sum_{t=1}^{T}\ddot{\boldsymbol{x}}_{t-1}\boldsymbol{z}_{2t}' = \sum_{t=1}^{T}\boldsymbol{\mu}_*(t-1)\boldsymbol{z}_{2t}' + \sum_{t=1}^{T}\mathbf{m}_{t-1}\boldsymbol{z}_{2t}'$. The proof of Lemma B.2 (*iii*) already shows that $T^{-2}\sum_{t=1}^{T}(t-1)\boldsymbol{z}_{2t} \xrightarrow{\mathbb{P}} \frac{1}{2}\mathbb{E}[\boldsymbol{z}_{2t}]$. Furthermore, $\sum_{t=1}^{T}\mathbf{m}_{t-1}\boldsymbol{z}_{2t}' = O_{\mathbb{P}}(T^{3/2})$ as shown in the proof of Lemma B.2 (*v*). Therefore, it follows that $T^{-2}\sum_{t=1}^{T}\ddot{\boldsymbol{x}}_{t-1}\boldsymbol{z}_{2t}' \xrightarrow{\mathbb{P}} \frac{1}{2}\boldsymbol{\mu}_*\mathbb{E}[\boldsymbol{z}_{2t}']$, as desired.

(*v*) Note that $\ddot{\mathbf{m}}_{t-1} = \mathbf{m}_{t-1} - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)(t-1)$. Therefore, $\sum_{t=1}^{T}\ddot{\mathbf{m}}_{t-1}\boldsymbol{z}_{2t}' = \sum_{t=1}^{T}\mathbf{m}_{t-1}\boldsymbol{z}_{2t}' - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)\sum_{t=1}^{T}(t-1)\boldsymbol{z}_{2t}'$. We here note that $\sum_{t=1}^{T}\mathbf{m}_{t-1}\boldsymbol{z}_{2t}' = \sum_{t=1}^{T}\mathbf{m}_{t-1}\mathbb{E}[\boldsymbol{z}_{2t}'] + \sum_{t=1}^{T}\mathbf{m}_{t-1}(\boldsymbol{z}_{2t}' - \mathbb{E}[\boldsymbol{z}_{2t}'])$. The proof of Lemma A.3 (*iv*) implies that $T^{-3/2}\sum_{t=1}^{T}\mathbf{m}_{t-1} \Rightarrow \int\boldsymbol{\mathscr{B}}_s$, and $\sum_{t=1}^{T}\mathbf{m}_{t-1}(\boldsymbol{z}_{2t}' - \mathbb{E}[\boldsymbol{z}_{2t}']) = o_{\mathbb{P}}(T^{3/2})$ by noting that $\mathbf{m}_{t-1} = O_{\mathbb{P}}(T^{1/2})$ and $\sum_{t=1}^{T}(\boldsymbol{z}_{2t}' - \mathbb{E}[\boldsymbol{z}_{2t}']) = O_{\mathbb{P}}(T^{1/2})$. Therefore, $T^{-3/2}\sum_{t=1}^{T}\mathbf{m}_{t-1}\boldsymbol{z}_{2t}' \Rightarrow \int\boldsymbol{\mathscr{B}}_s\mathbb{E}[\boldsymbol{z}_{2t}']$. Next, $T^{-3/2}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)\sum_{t=1}^{T}(t-1)\boldsymbol{z}_{2t}' = \sqrt{T}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)T^{-2}\sum_{t=1}^{T}(t-1)\boldsymbol{z}_{2t}'$, and Lemma B.7 (*i*) implies that $\sqrt{T}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*) \Rightarrow 3\int r\boldsymbol{\mathscr{B}}_s$. In addition to this, Lemma B.2 (*iii*) shows that $T^{-2}\sum_{t=1}^{T}(t-1)\boldsymbol{z}_{2t}' \xrightarrow{\mathbb{P}} \frac{1}{2}\mathbb{E}[\boldsymbol{z}_{2t}']$. Therefore, $T^{-3/2}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)\sum_{t=1}^{T}(t-1)\boldsymbol{z}_{2t}' \Rightarrow \frac{3}{2}\int r\boldsymbol{\mathscr{B}}_s\mathbb{E}[\boldsymbol{z}_{2t}']$. Hence, if we combine all these, $T^{-3/2}\sum_{t=1}^{T}\ddot{\mathbf{m}}_{t-1}\boldsymbol{z}_{2t}' \Rightarrow (\int\boldsymbol{\mathscr{B}}_s - \frac{3}{2}\int r\boldsymbol{\mathscr{B}}_s)\mathbb{E}[\boldsymbol{z}_{2t}'] = \int\widetilde{\boldsymbol{\mathscr{B}}}_s\mathbb{E}[\boldsymbol{z}_{2t}']$.

(*vi*) Note that $y_{t-1} = \delta_*(t-1) + \sum_{j=1}^{t-1}d_j$. Therefore, $\sum_{t=1}^{T}y_{t-1}\boldsymbol{z}_{2t}' = \sum_{t=1}^{T}\delta_*(t-1)\boldsymbol{z}_{2t}' + \sum_{t=1}^{T}(\sum_{j=1}^{t-1}d_j)\boldsymbol{z}_{2t}'$. The proof of Lemma B.2 (*iii*) already showed that $T^{-2}\sum_{t=1}^{T}(t-1)\boldsymbol{z}_{2t} \xrightarrow{\mathbb{P}} \frac{1}{2}\mathbb{E}[\boldsymbol{z}_{2t}]$. Furthermore, we note that $\sum_{t=1}^{T}(\sum_{j=1}^{t-1}d_j)\boldsymbol{z}_{2t}' = O_{\mathbb{P}}(T^{3/2})$. Therefore, $T^{-2}\sum_{t=1}^{T}y_{t-1}\boldsymbol{z}_{2t}' \xrightarrow{\mathbb{P}} \frac{1}{2}\delta_*\mathbb{E}[\boldsymbol{z}_{2t}]$, as desired.

(*vii*) In the proof of Lemma B.6 (*iv*), we show that $\ddot{u}_t = u_t + O_{\mathbb{P}}(T^{-1/2})$. Therefore, it follows that $T^{-1}\sum_{t=1}^{T}\ddot{u}_{t-1}\boldsymbol{z}_{2t} = T^{-1}\sum_{t=1}^{T}u_{t-1}\boldsymbol{z}_{2t} + o_{\mathbb{P}}(1) \xrightarrow{\mathbb{P}} \mathbb{E}[u_{t-1}\boldsymbol{z}_{2t}]$ by the ergodic theorem. This completes the proof. ∎

Proof of Lemma B.3. (*i*) $\sum_{t=1}^{T}t = T(T+1)/2$, leading to the desired result.

(*ii*) Note that $\ddot{\boldsymbol{x}}_t = \boldsymbol{\mu}_*t + \mathbf{m}_t$, so that $\sum_{t=1}^{T}\ddot{\boldsymbol{x}}_t = \boldsymbol{\mu}_*\sum_{t=1}^{T}t + \sum_{t=1}^{T}\mathbf{m}_t$. Here, $T^{-2}\sum_{t=1}^{T}t \rightarrow \frac{1}{2}$ by Lemma B.3 (*i*), and Lemma B.1 implies that $T^{-3/2}\sum_{t=1}^{T}\mathbf{m}_t = T^{-1}\sum_{t=1}^{T}\frac{1}{\sqrt{T}}\mathbf{m}_t = \int\mathbf{B}_s(r)dr \Rightarrow \int\boldsymbol{\mathscr{B}}_s$. Therefore, $T^{-3/2}\sum_{t=1}^{T}\ddot{\boldsymbol{x}}_t = \boldsymbol{\mu}_*T^{-3/2}\sum_{t=1}^{T}t + o_{\mathbb{P}}(1) \xrightarrow{\mathbb{P}} \frac{1}{2}\boldsymbol{\mu}_*$.

(*iii*) Note that $\ddot{\mathbf{m}}_t = \mathbf{m}_t - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)t$. Thus, $T^{-3/2}\sum_{t=1}^{T}\ddot{\mathbf{m}}_t = T^{-3/2}\sum_{t=1}^{T}\mathbf{m}_t - \sqrt{T}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)T^{-2}\sum_{t=1}^{T}t$. Lemma B.7 (*i*) implies that $\sqrt{T}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*) \Rightarrow 3\int r\boldsymbol{\mathscr{B}}_s$. Lemma B.3 (*i*) implies that $T^{-2}\sum_{t=1}^{T}t \rightarrow \frac{1}{2}$. In addition to these, $T^{-3/2}\sum_{t=1}^{T}\mathbf{m}_t = T^{-1}\sum_{t=1}^{T}\frac{1}{\sqrt{T}}\mathbf{m}_t = \int\mathbf{B}_s(r)dr \Rightarrow \int\boldsymbol{\mathscr{B}}_s$ by Lemma B.1. Thus, $T^{-3/2}\sum_{t=1}^{T}\ddot{\mathbf{m}}_t \Rightarrow \int\boldsymbol{\mathscr{B}}_s(r) - 3\int rdr\int v\boldsymbol{\mathscr{B}}_s(v)dv = \int\widetilde{\boldsymbol{\mathscr{B}}}_s$ by the definition of $\widetilde{\boldsymbol{\mathscr{B}}}_s(\cdot)$, viz., $\widetilde{\boldsymbol{\mathscr{B}}}_s(\cdot) := \boldsymbol{\mathscr{B}}_s(\cdot) - \frac{3}{2}(\cdot)\int v\boldsymbol{\mathscr{B}}_s(v)dv$.

(*iv*) From (5), $T^{-2}\sum_{t=1}^{T}y_t = \delta_*T^{-2}\sum_{t=1}^{T}t + T^{-2}\sum_{t=1}^{T}\sum_{j=1}^{t}d_j = \frac{1}{2}\delta_* + o_{\mathbb{P}}(1)$.

(v) We note that $\ddot{u}_{t-1} = \widetilde{u}_{t-1} := y_{t-1} - \boldsymbol{r}'_{t-1}\widetilde{\boldsymbol{\upsilon}}_T$, so that $\sum_{t=1}^T \ddot{u}_{t-1}\boldsymbol{r}_{t-1} \equiv \mathbf{0}$. We here note that $\boldsymbol{r}_{t-1} := [\ddot{\boldsymbol{x}}'_{t-1}, (t-1), 1]'$. This completes the proof. ■

Proof of Lemma B.4. (i) $\sum_{t=1}^T t^2 = T(T+1)(2T+1)/6$, leading to the desired result.

(ii) $T^{-3}\sum_{t=1}^T t\ddot{\boldsymbol{x}}_t = \boldsymbol{\mu}_* T^{-3}\sum_{t=1}^T t^2 + T^{-3}\sum_{t=1}^T t\mathbf{m}_t$. Here, it follows that $T^{-3}\sum_{t=1}^T t^2 \to \frac{1}{3}$ and $T^{-5/2}\sum_{t=1}^T t\mathbf{m}_t = T^{-1}\sum_{t=1}^T(\frac{t}{T})\frac{1}{\sqrt{T}}\mathbf{m}_t = \int r\mathbf{B}_{sT}(r)dr \Rightarrow \int r\boldsymbol{\mathscr{B}}_s$. Therefore, $T^{-3}\sum_{t=1}^T t\ddot{\boldsymbol{x}}_t = \frac{1}{3}\boldsymbol{\mu}_* + o_{\mathbb{P}}(1)$.

(iii) Note that $\ddot{\mathbf{m}}_t = \ddot{\boldsymbol{x}}_t - t\ddot{\boldsymbol{\mu}}_T$ and $\ddot{\boldsymbol{\mu}}_T = (\sum_{t=1}^{T-1} t^2)^{-1}\sum_{t=1}^{T-1} t\ddot{\boldsymbol{x}}_t$. Therefore, $\sum_{t=1}^T (t-1)\ddot{\mathbf{m}}_{t-1} \equiv \mathbf{0}$.

(iv) From (5), $T^{-3}\sum_{t=1}^T ty_t = \delta_* T^{-3}\sum_{t=1}^T t^2 + T^{-3}\sum_{t=1}^T t\sum_{j=1}^t d_j = \frac{1}{3}\delta_* + o_{\mathbb{P}}(1)$.

(v) The proof of Lemma B.3 (v) already shows the given claim. ■

Proof of Lemma B.5. (i) Using the fact that $\ddot{\boldsymbol{x}}_t = \boldsymbol{\mu}_* t + \mathbf{m}_t$, $\sum_{t=1}^T \ddot{\boldsymbol{x}}_t\ddot{\boldsymbol{x}}'_t = \sum_{t=1}^T(\boldsymbol{\mu}_* t + \mathbf{m}_t)(\boldsymbol{\mu}_* t + \mathbf{m}_t)' = \sum_{t=1}^T \boldsymbol{\mu}_*\boldsymbol{\mu}'_* t^2 + \sum_{t=1}^T \boldsymbol{\mu}_*\mathbf{m}'_t t + \sum_{t=1}^T \mathbf{m}_t\boldsymbol{\mu}'_* t + \sum_{t=1}^T \mathbf{m}_t\mathbf{m}'_t$. Here, $T^{-3}\sum_{t=1}^T \boldsymbol{\mu}_*\boldsymbol{\mu}'_* t^2 \to \frac{1}{3}\boldsymbol{\mu}_*\boldsymbol{\mu}'_*$ by Lemma B.4 (i), and $T^{-5/2}\sum_{t=1}^T \boldsymbol{\mu}_*\mathbf{m}'_t t \Rightarrow \boldsymbol{\mu}_*\int r\boldsymbol{\mathscr{B}}_s$ as shown in the proof of Lemma B.4 (ii). Furthermore, $T^{-2}\sum_{t=1}^T \mathbf{m}_t\mathbf{m}'_t = T^{-1}\sum_{t=1}^T \frac{1}{\sqrt{T}}\mathbf{m}_t\frac{1}{\sqrt{T}}\mathbf{m}'_t = \int \mathbf{B}_{sT}(r)\mathbf{B}_{sT}(r)'dr \Rightarrow \int \boldsymbol{\mathscr{B}}_s\boldsymbol{\mathscr{B}}'_s$. Thus, $T^{-3}\sum_{t=1}^T \ddot{\boldsymbol{x}}_t\ddot{\boldsymbol{x}}'_t \xrightarrow{\mathbb{P}} \frac{1}{3}\boldsymbol{\mu}_*\boldsymbol{\mu}'_*$.

(ii) We first note that $\sum_{t=1}^T \ddot{\boldsymbol{x}}_{t-1}\ddot{\mathbf{m}}'_{t-1} = \sum_{t=1}^T(\boldsymbol{\mu}_*(t-1) + \mathbf{m}_{t-1})\ddot{\mathbf{m}}'_{t-1} = \boldsymbol{\mu}_*\sum_{t=1}^T(t-1)\ddot{\mathbf{m}}'_{t-1} + \sum_{t=1}^T \mathbf{m}_{t-1}\ddot{\mathbf{m}}'_{t-1} = \sum_{t=1}^T \mathbf{m}_{t-1}\ddot{\mathbf{m}}'_{t-1}$ by Lemma B.4 (iii). We further note that $\sum_{t=1}^T \mathbf{m}_{t-1}\ddot{\mathbf{m}}'_{t-1} = \sum_{t=1}^T \mathbf{m}_{t-1}(\mathbf{m}_{t-1} - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)(t-1))'$ using the fact that $\ddot{\mathbf{m}}_t = \mathbf{m}_t - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)t$. Here, we note that $T^{-2}\sum_{t=1}^T \mathbf{m}_t\mathbf{m}'_t \Rightarrow \int \boldsymbol{\mathscr{B}}_s\boldsymbol{\mathscr{B}}'_s$ as shown in the proof of Lemma B.5 (i), and $T^{-2}\sum_{t=1}^T t\mathbf{m}_t(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)' = T^{-5/2}\sum_{t=1}^T t\mathbf{m}_t\sqrt{T}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)' \Rightarrow 3\int r\boldsymbol{\mathscr{B}}_s\int r\boldsymbol{\mathscr{B}}_s$ by Lemma B.7 (ii) and the fact that $T^{-5/2}\sum_{t=1}^T t\mathbf{m}_t = T^{-1}\sum_{t=1}^T \frac{t}{T}\frac{1}{\sqrt{T}}\mathbf{m}_t = \int r\mathbf{B}_s(r)dr \Rightarrow \int r\boldsymbol{\mathscr{B}}_s$ as shown in the proof of Lemma B.4 (ii). Therefore, $T^{-2}\sum_{t=1}^T \ddot{\boldsymbol{x}}_{t-1}\ddot{\mathbf{m}}'_{t-1} \Rightarrow \int \boldsymbol{\mathscr{B}}_s(\boldsymbol{\mathscr{B}}_s - 3r\int s\boldsymbol{\mathscr{B}}_s)' = \int \boldsymbol{\mathscr{B}}_s\bar{\boldsymbol{\mathscr{B}}}'_s$. We further note that $\int \boldsymbol{\mathscr{B}}_s\bar{\boldsymbol{\mathscr{B}}}'_s = \int \bar{\boldsymbol{\mathscr{B}}}_s\bar{\boldsymbol{\mathscr{B}}}'_s$ by the definition of $\bar{\boldsymbol{\mathscr{B}}}_s$ and the fact that $\int r^2 = \frac{1}{3}$.

(iii) Using the fact that $\ddot{\boldsymbol{x}}_t = \boldsymbol{\mu}_* t + \mathbf{m}_t$ and $y_t = \delta_* t + \sum_{j=1}^t d_j$, $\sum_{t=1}^T \ddot{\boldsymbol{x}}_t y_t = \boldsymbol{\mu}_*\delta_*\sum_{t=1}^T t^2 + \delta_*\sum_{t=1}^T t\mathbf{m}_t + \boldsymbol{\mu}_*\sum_{t=1}^T t\sum_{j=1}^t d_j + \sum_{t=1}^T \mathbf{m}_t\sum_{j=1}^T d_j$. Here, $T^{-3}\sum_{t=1}^T t^2 \to \frac{1}{3}$ by Lemma B.4 (i), and $T^{-5/2}\sum_{t=1}^T t\mathbf{m}_t \Rightarrow \int r\boldsymbol{\mathscr{B}}_s$ as shown in the proof of Lemma B.4 (ii). Furthermore, we note that $\sum_{t=1}^T t\sum_{j=1}^t d_j = O_{\mathbb{P}}(T^{5/2})$ and $\sum_{t=1}^T \mathbf{m}_t\sum_{j=1}^T d_j = O_{\mathbb{P}}(T^2)$. Therefore, $T^{-3}\sum_{t=1}^T \ddot{\boldsymbol{x}}_t y_t \xrightarrow{\mathbb{P}} \frac{1}{3}\delta_*\boldsymbol{\mu}_*$.

(iv) The proof of Lemma B.3 (v) already shows the given claim. ■

Proof of Lemma B.6. (i) We first note that $\sum_{t=1}^T \ddot{\mathbf{m}}_{t-1}\ddot{\mathbf{m}}'_{t-1} = \sum_{t=1}^T(\mathbf{m}_{t-1} - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)(t-1))\ddot{\mathbf{m}}'_{t-1} = \sum_{t=1}^T \mathbf{m}_{t-1}\ddot{\mathbf{m}}'_{t-1} - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)\sum_{t=1}^T(t-1)\ddot{\mathbf{m}}'_{t-1} = \sum_{t=1}^T \mathbf{m}_{t-1}\ddot{\mathbf{m}}'_{t-1}$ by Lemma B.4 (iii). We next note that $\sum_{t=1}^T \mathbf{m}_{t-1}\ddot{\mathbf{m}}'_{t-1} = \sum_{t=1}^T \mathbf{m}_{t-1}(\mathbf{m}_{t-1} - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)(t-1))'$ using the fact that $\ddot{\mathbf{m}}_t = \mathbf{m}_t - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)t$. Here, $T^{-2}\sum_{t=1}^T \mathbf{m}_t\mathbf{m}'_t \Rightarrow \int \boldsymbol{\mathscr{B}}_s\boldsymbol{\mathscr{B}}'_s$ as shown in the proof of Lemma B.5 (i), and $T^{-2}\sum_{t=1}^T t\mathbf{m}_t(\ddot{\boldsymbol{\mu}}_T -$

$\boldsymbol{\mu}_*)' \Rightarrow 3\int r\mathscr{B}_s \int r\mathscr{B}_s$ as shown in the proof of Lemma B.5 (*ii*). Therefore, $T^{-2}\sum_{t=1}^T \ddot{\mathbf{m}}_{t-1}\ddot{\mathbf{m}}_{t-1}' \Rightarrow \int \mathscr{B}_s(\mathscr{B}_s - 3r\int s\mathscr{B}_s)' = \int \mathscr{B}_s\bar{\mathscr{B}}_s'$.

(*ii*) As Lemma B.7 (*iii*) shows, for each t, $\boldsymbol{r}_t'\widetilde{\boldsymbol{\upsilon}}_T = \ddot{\boldsymbol{r}}_t'\ddot{\boldsymbol{\upsilon}}_T$, so that $\widetilde{u}_t := y_t - \boldsymbol{r}_t'\widetilde{\boldsymbol{\upsilon}}_T = \ddot{u}_{t-1} := y_t - \ddot{\boldsymbol{r}}_t'\ddot{\boldsymbol{\upsilon}}_T$. We further note that $\ddot{\boldsymbol{\upsilon}}_T := (\sum_{t=1}^T \ddot{\boldsymbol{r}}_{t-1}\ddot{\boldsymbol{r}}_{t-1}')^{-1}\sum_{t=1}^T \ddot{\boldsymbol{r}}_{t-1}y_{t-1}$, so that $\sum_{t=1}^T \ddot{\boldsymbol{r}}_{t-1}\widetilde{u}_{t-1} = \mathbf{0}$. We now note that $\ddot{\boldsymbol{r}}_{t-1} := [\ddot{\mathbf{m}}_{t-1}', (t-1), 1]'$, leading to that $\sum_{t=1}^T \ddot{\mathbf{m}}_{t-1}\widetilde{u}_{t-1} = \mathbf{0}$.

(*iii*) From (5), $y_t = \delta_* t + \sum_{j=1}^t d_j$. We here note that $\sum_{t=1}^T y_t^2 = \sum_{t=1}^T(\delta_* t + \sum_{j=1}^t d_j)^2 = \sum_{t=1}^T \delta_*^2 t^2 + 2\sum_{t=1}^T \delta_* t\sum_{j=1}^t d_j + \sum_{t=1}^T(\sum_{j=1}^t d_j)^2$. Furthermore, $T^{-3}\sum_{t=1}^T t^2 \to \frac{1}{3}$ by Lemma B.4 (*i*), $\sum_{t=1}^T \delta_* t\sum_{j=1}^t d_j = O_{\mathbb{P}}(T^{5/2})$, and $\sum_{t=1}^T(\sum_{j=1}^t d_j)^2 = O_{\mathbb{P}}(T^2)$, implying that $T^{-3}\sum_{t=1}^T y_t^2 \overset{\mathbb{P}}{\to} \frac{1}{3}\delta_*^2$.

(*iv*) Note that $\widetilde{u}_t := y_t - \boldsymbol{r}_t'\widetilde{\boldsymbol{\upsilon}}_T$, and $\boldsymbol{r}_t'\widetilde{\boldsymbol{\upsilon}}_T = \ddot{\boldsymbol{x}}_t'\widetilde{\boldsymbol{\beta}}_T + t\widetilde{\zeta}_T + \widetilde{\nu}_T$ with $\ddot{\boldsymbol{x}}_t = \ddot{\mathbf{m}}_t + \ddot{\boldsymbol{\mu}}_T t$. Furthermore, $y_t = \boldsymbol{\beta}_*'(\ddot{\mathbf{m}}_t + \ddot{\boldsymbol{\mu}}_T t) + \zeta_* t + \nu_* + u_t$ using (6). Hence, $\widetilde{u}_t = u_t - (\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*)'\ddot{\mathbf{m}}_t - (\ddot{\vartheta}_T - \vartheta_{T*})t - (\ddot{\nu}_T - \nu_*)$. We now note that Lemma 3, $\ddot{\mathbf{m}}_t = O_{\mathbb{P}}(T^{1/2})$, and $t = O(T)$ imply that $\widetilde{u}_t = u_t + O_{\mathbb{P}}(T^{-1/2})$. Therefore, $T^{-1}\sum_{t=1}^T \widetilde{u}_t^2 = T^{-1}\sum_{t=1}^T u_t^2 + o_{\mathbb{P}}(1)$, and $T^{-1}\sum_{t=1}^T u_t^2 \overset{\mathbb{P}}{\to} \mathbb{E}[u_t^2]$ by the ergodic theorem, implying that $T^{-1}\sum_{t=1}^T \widetilde{u}_t^2 \overset{\mathbb{P}}{\to} \sigma_u^2 := \mathbb{E}[u_t^2]$. ■

Proof of Lemma B.7. (*i*) Note that $\ddot{\boldsymbol{\mu}}_T = (\sum_{t=1}^{T-1} t^2)^{-1}\sum_{t=1}^{T-1} t\ddot{\boldsymbol{x}}_t$ and $\ddot{\boldsymbol{x}}_t = \boldsymbol{\mu}_* t + \mathbf{m}_t$. Therefore, $\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_* = (\sum_{t=1}^{T-1} t^2)^{-1}\sum_{t=1}^{T-1} t\mathbf{m}_t$, so that $\sqrt{T}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*) = (T^{-3}\sum_{t=1}^{T-1} t^2)^{-1}T^{-5/2}\sum_{t=1}^{T-1} t\mathbf{m}_t$. Lemma B.4 (*i*) implies that $T^{-3}\sum_{t=1}^T t^2 \to \frac{1}{3}$. Furthermore, $T^{-5/2}\sum_{t=1}^{T-1} t\mathbf{m}_t = T^{-1}\sum_{t=1}^{T-1} \frac{t}{T}\frac{1}{\sqrt{T}}\mathbf{m}_t = \int r\mathbf{B}_{sT}(r)dr \Rightarrow \int r\mathscr{B}_s$. Hence, $\sqrt{T}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*) \Rightarrow 3\int r\mathscr{B}_s$.

(*ii*) Note that $\boldsymbol{r}_t'\widetilde{\boldsymbol{\upsilon}}_T = \ddot{\boldsymbol{x}}_t'\widetilde{\boldsymbol{\beta}}_T + t\widetilde{\zeta}_T + \widetilde{\nu}_T$ with $\ddot{\boldsymbol{x}}_t = \ddot{\mathbf{m}}_t + \ddot{\boldsymbol{\mu}}_T t$. Therefore, $\boldsymbol{r}_t'\widetilde{\boldsymbol{\upsilon}}_T = \ddot{\mathbf{m}}_t'\widetilde{\boldsymbol{\beta}}_T + t(\ddot{\boldsymbol{\mu}}_T'\widetilde{\boldsymbol{\beta}}_T + \widetilde{\zeta}_T) + \widetilde{\nu}_T = \ddot{\mathbf{m}}_t'\ddot{\boldsymbol{\beta}}_T + t\ddot{\vartheta}_T + \ddot{\nu}_T = \ddot{\boldsymbol{r}}_t'\ddot{\boldsymbol{\upsilon}}_T$, where the second last equality holds by Proposition 2, and the last equality follows from the definition of $\ddot{\boldsymbol{r}}_t$.

(*iii*) Note that $T^{-1}\sum_{t=1}^T \mathbf{m}_{t-1}u_t = T^{-1}\sum_{t=1}^T T^{-1/2}\mathbf{m}_{t-1}\sqrt{T}(B_{uT}(t/T) - B_{uT}((t-1)/T)) = \int \mathbf{B}_{sT}(r)dB_{uT}(r)$. We here note that $\int \mathbf{B}_{sT}(r)dB_{uT}(r) \Rightarrow \int \mathscr{B}_s d\mathscr{B}_u + \boldsymbol{\varrho}_{m*}$ by applying theorem 4 of de Jong and Davidson (2000).

(*iv*) We first note that $T^{-1}\sum_{t=1}^T \mathbf{m}_{t-1}e_t = T^{-1}\sum_{t=1}^T T^{-1/2}\mathbf{m}_{t-1}\sqrt{T}(B_{eT}(t/T) - B_{eT}((t-1)/T)) = \int \mathbf{B}_{sT}(r)dB_{eT}(r)$. Note that $\int \mathbf{B}_{sT}(r)dB_{eT}(r) \Rightarrow \int \mathscr{B}_s d\mathscr{B}_e$ by applying theorem 4 of de Jong and Davidson (2000) and noting that $\mathbb{E}[\boldsymbol{s}_\tau e_t] = \mathbf{0}$ for each $\tau < t$.

(*v*) Note that if we let $\ddot{e}_t := \Delta y_t - \ddot{\boldsymbol{z}}_t\ddot{\boldsymbol{\tau}}_T$, it follows that $\widehat{\sigma}_{e,T}^2 = T^{-1}\sum_{t=1}^T \ddot{e}_t^2$ and that $\ddot{e}_t = -\ddot{\boldsymbol{z}}_t'(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) + e_t$ from the fact that $\Delta y_t = \ddot{\boldsymbol{z}}_t'\boldsymbol{\tau}_{T*} + e_t$, implying that

$$\sum_{t=1}^T \ddot{e}_t^2 = (\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*})'\ddot{\mathbf{D}}\left(\ddot{\mathbf{D}}^{-1}\sum_{t=1}^T \ddot{\boldsymbol{z}}_t\ddot{\boldsymbol{z}}_t'\ddot{\mathbf{D}}^{-1}\right)\ddot{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) - 2\left(\sum_{t=1}^T e_t\ddot{\boldsymbol{z}}_t\right)\ddot{\mathbf{D}}^{-1}\ddot{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) + \sum_{t=1}^T e_t^2.$$

We examine the asymptotic behavior of each element on the right side. First, Lemmas 3 (*vi*) and 4 (*i*) imply that $\ddot{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) = O_{\mathbb{P}}(1)$ and $\ddot{\mathbf{D}}^{-1}\sum_{t=1}^T \ddot{\boldsymbol{z}}_t\ddot{\boldsymbol{z}}_t'\ddot{\mathbf{D}}^{-1} = O_{\mathbb{P}}(1)$. Second, from the definitions of $\ddot{\boldsymbol{z}}_t$ and $\ddot{\mathbf{D}}$,

$\sum_{t=1}^T e_t\ddot{\boldsymbol{z}}_t\ddot{\mathbf{D}}^{-1} = [T^{-1/2}\sum_{t=1}^T e_t\ddot{u}_{t-1}, T^{-1}\sum_{t=1}^T e_t\ddot{\mathbf{m}}_{t-1}', T^{-3/2}\sum_{t=1}^T e_t(t-1), T^{-1/2}\sum_{t=1}^T e_t, T^{-1/2}\sum_{t=1}^T e_t\boldsymbol{z}_{2t}']'$. We verify that each element on the right side is $O_{\mathbb{P}}(1)$. We first note that $T^{-3/2}\sum_{t=1}^T e_t(t-1) = O_{\mathbb{P}}(1)$, $T^{-1/2}\sum_{t=1}^T e_t = O_{\mathbb{P}}(1)$, and $T^{-1/2}\sum_{t=1}^T e_t\boldsymbol{z}_{2t} = O_{\mathbb{P}}(1)$ by the martingale difference CLT based upon the fact that $\{e_t, \mathscr{F}_t\}$ is an MDA. In addition, $T^{-1}\sum_{t=1}^T e_t\ddot{\mathbf{m}}_{t-1} = T^{-1}\sum_{t=1}^T e_t\mathbf{m}_{t-1} - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)T^{-1}\sum_{t=1}^T e_t(t-1)$ by noting that $\ddot{\mathbf{m}}_{t-1} = \mathbf{m}_{t-1} - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)(t-1)$ as given in the proof of Lemma B.6 (*i*). Here, Lemmas B.7 (*i* and *iv*) imply that $(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*) = O_{\mathbb{P}}(T^{-1/2})$ and $T^{-1}\sum_{t=1}^T e_t\mathbf{m}_{t-1} = O_{\mathbb{P}}(1)$, respectively, so that $T^{-1}\sum_{t=1}^T e_t\ddot{\mathbf{m}}_{t-1} = O_{\mathbb{P}}(1)$. Finally, $T^{-1/2}\sum_{t=1}^T e_t\ddot{u}_{t-1} = T^{-1/2}\sum_{t=1}^T e_t u_{t-1} + o_{\mathbb{P}}(1)$ using the fact that $\ddot{u}_{t-1} = u_{t-1} + O_{\mathbb{P}}(T^{-1/2})$ as given in the proof of Lemma B.6 (*iv*). All these facts imply that $\sum_{t=1}^T e_t\ddot{\boldsymbol{z}}_t\ddot{\mathbf{D}}^{-1} = O_{\mathbb{P}}(1)$. By these two facts, it follows that $\sum_{t=1}^T \ddot{e}_t^2 = \sum_{t=1}^T e_t^2 + O_{\mathbb{P}}(1)$, implying that $\widehat{\sigma}_{e,T}^2 := T^{-1}\sum_{t=1}^T \ddot{e}_t^2 = T^{-1}\sum_{t=1}^T e_t^2 + O_{\mathbb{P}}(T^{-1})$. The desired result follows from the ergodic theorem, and this completes the proof. ■

A.3 Proofs

Proof of Lemma 1. (*i*) This follows from Lemmas B.3 (*i*, *ii*, *iv*), B.4 (*i*, *ii*, *iv*), B.5 (*i*, *iii*), B.6 (*iii*), and the remarks below Lemmas B.3, B.4, and B.5.

(*ii*) This follows from Lemma B.2 (*i*).

(*iii*) This follows from Lemmas B.2 (*ii*, *iii*, *iv*, *vi*) and the remark below Lemma B.2.

(*iv*) This follows from Lemmas 1 (*i*, *ii*, *iii*) and the structure of $\mathbf{M}$. ■

Proof of Lemma 2. (*i*) This follows from Lemmas B.3 (*v*), B.4 (*v*), and B.6 (*ii*).

(*ii*) This follows from Lemmas B.3 (*i*, *ii*, *iii*), B.4 (*i*, *ii*, *iii*), and B.6 (*i*).

(*iii*) This follows from Lemma B.2 (*vii*).

(*iv*) This follows from Lemmas B.2 (*ii*, *iii*, *iv*, and *vi*).

(*v*) This follows from Lemma B.2 (*i*).

(*vi*) This follows from Lemma A.3 (*i*) and Lemmas 2 (*i*, *ii*, *iii*, *iv*, *v*). ■

Proof of Lemma 3. (*i*) We note that $\ddot{\mathbf{D}}_1(\ddot{\boldsymbol{v}}_T - \bar{\boldsymbol{v}}_{T*}) = (\ddot{\mathbf{D}}_1^{-1}\sum_{t=1}^T \ddot{\boldsymbol{r}}_{t-1}\ddot{\boldsymbol{r}}_{t-1}'\ddot{\mathbf{D}}_1^{-1})^{-1}\ddot{\mathbf{D}}_1^{-1}\sum_{t=1}^T \ddot{\boldsymbol{r}}_{t-1}u_{t-1}$ from (13), and Lemma 2 (*vii*) implies that $\ddot{\mathbf{D}}_1^{-1}(\sum_{t=1}^T \ddot{\boldsymbol{r}}_{t-1}\ddot{\boldsymbol{r}}_{t-1}')\ddot{\mathbf{D}}_1^{-1} \Rightarrow \mathscr{M}_{11}$. We therefore focus on the limit distribution of $\ddot{\mathbf{D}}_1^{-1}\sum_{t=1}^T \ddot{\boldsymbol{r}}_{t-1}u_{t-1}$. Note that

$$\ddot{\mathbf{D}}_1^{-1}\sum_{t=1}^T \ddot{\boldsymbol{r}}_{t-1}u_{t-1} = \left[T^{-1}\sum_{t=1}^T \ddot{\mathbf{m}}_{t-1}'u_{t-1}, T^{-3/2}\sum_{t=1}^T (t-1)u_{t-1}, T^{-1/2}\sum_{t=1}^T u_{t-1}\right]'.$$

We now examine the asymptotic behavior of each element on the right side. First, we note that $\ddot{\mathbf{m}}_{t-1} = \mathbf{m}_{t-1} - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)(t-1)$. Therefore, $T^{-1}\sum_{t=1}^T \ddot{\mathbf{m}}_{t-1}u_{t-1} = T^{-1}\sum_{t=1}^T \mathbf{m}_{t-2}u_{t-1} + T^{-1}\sum_{t=1}^T \boldsymbol{s}_{t-1}u_{t-1} -$

$(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)T^{-1}\sum_{t=1}^T (t-1)u_{t-1}$. We here note that $T^{-1}\sum_{t=1}^T \mathbf{m}_{t-2}u_{t-1} \Rightarrow \int \boldsymbol{\mathscr{B}}_s d\mathscr{B}_u + \boldsymbol{\varrho}_{m*}$ by Lemma B.7 (*iii*), and $\sqrt{T}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*) \Rightarrow 3\int r\boldsymbol{\mathscr{B}}_s$ by Lemma B.7 (*i*). In addition to this, $T^{-3/2}\sum_{t=1}^T (t-1)u_{t-1} = T^{-1}\sum_{t=1}^T \frac{(t-1)}{T}\sqrt{T}(B_{uT}((t-1)/T) - B_{uT}((t-2)/T)) = \int r dB_{uT}(r) \Rightarrow \int r d\mathscr{B}_u$. Hence, it follows that $T^{-1}\sum_{t=1}^T \ddot{\mathbf{m}}_{t-1}u_{t-1} \Rightarrow \boldsymbol{\mathscr{S}}_1 := \int \boldsymbol{\mathscr{B}}_s d\mathscr{B}_u + \boldsymbol{\varrho}_{m*} - 3\int r\boldsymbol{\mathscr{B}}_s \int r d\mathscr{B}_u$. Next, it is already showed that $T^{-3/2}\sum_{t=1}^T (t-1)u_{t-1} \Rightarrow \mathscr{S}_2 := \int r d\mathscr{B}_u$. Third, note that $T^{-1/2}\sum_{t=1}^T u_{t-1} = T^{-1}\sum_{t=1}^T \sqrt{T}(B_{uT}((t-1)/T) - B_{uT}((t-2)/T)) = \int r dB_{uT}(r) \Rightarrow \mathscr{S}_3 := \int d\mathscr{B}_u$. We now combine the first to third facts to obtain that $\ddot{\mathbf{D}}_1^{-1}\sum_{t=1}^T \ddot{\boldsymbol{r}}_{t-1}u_{t-1} \Rightarrow \boldsymbol{\mathscr{S}}$, leading to that $\ddot{\mathbf{D}}_1(\ddot{\boldsymbol{v}}_T - \bar{\boldsymbol{v}}_{T*}) \Rightarrow \boldsymbol{\mathscr{L}} := \boldsymbol{\mathscr{M}}_{11}^{-1}\boldsymbol{\mathscr{S}}$, as desired.

(*ii*) We first note that $\widetilde{\boldsymbol{v}}_T - \boldsymbol{v}_* = \mathbf{P}_T\ddot{\boldsymbol{v}}_T - \mathbf{P}\bar{\boldsymbol{v}}_*$. Therefore, $\widetilde{\boldsymbol{v}}_T - \boldsymbol{v}_* = (\mathbf{P}_T - \mathbf{P})(\ddot{\boldsymbol{v}}_T - \bar{\boldsymbol{v}}_{T*}) + \mathbf{P}(\ddot{\boldsymbol{v}}_T - \bar{\boldsymbol{v}}_{T*}) + (\mathbf{P}_T - \mathbf{P})(\bar{\boldsymbol{v}}_{T*} - \bar{\boldsymbol{v}}_*) + \mathbf{P}(\bar{\boldsymbol{v}}_{T*} - \bar{\boldsymbol{v}}_*) + (\mathbf{P}_T - \mathbf{P})\bar{\boldsymbol{v}}_*$. From the definition of $\mathbf{P}_T$ and Lemma 3 (*i*), we note that $(\mathbf{P}_T - \mathbf{P}) = O_{\mathbb{P}}(T^{-1/2})$, $(\ddot{\boldsymbol{v}}_T - \bar{\boldsymbol{v}}_{T*}) = O_{\mathbb{P}}(\ddot{\mathbf{D}}^{-1})$, and $(\mathbf{P}_T - \mathbf{P})(\bar{\boldsymbol{v}}_{T*} - \bar{\boldsymbol{v}}_*) = \mathbf{0}$. Furthermore, $\mathbf{P}(\bar{\boldsymbol{v}}_{T*} - \bar{\boldsymbol{v}}_*) = [\mathbf{0}', \mathbf{0}', (\vartheta_{T*} - \vartheta_*), 0]'$ such that $(\vartheta_{T*} - \vartheta_*) = \boldsymbol{\beta}_*'(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)$, and $(\mathbf{P}_T - \mathbf{P})\bar{\boldsymbol{v}}_* = [\mathbf{0}, \mathbf{0}, -\boldsymbol{\beta}_*'(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*), 0]'$, so that $\mathbf{P}(\bar{\boldsymbol{v}}_{T*} - \bar{\boldsymbol{v}}_*) + (\mathbf{P}_T - \mathbf{P})\bar{\boldsymbol{v}}_* = \mathbf{0}$. Hence, it now follows that $\widetilde{\boldsymbol{v}}_T - \boldsymbol{v}_* = (\mathbf{P}_T - \mathbf{P})(\ddot{\boldsymbol{v}}_T - \bar{\boldsymbol{v}}_{T*}) + \mathbf{P}(\ddot{\boldsymbol{v}}_T - \bar{\boldsymbol{v}}_{T*}) = \mathbf{P}(\ddot{\boldsymbol{v}}_T - \bar{\boldsymbol{v}}_{T*}) + O_{\mathbb{P}}(T^{-3/2})$, so that

$$\ddot{\mathbf{D}}_{\dagger}(\widetilde{\boldsymbol{v}}_T - \boldsymbol{v}_*) = [\; T(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*)' \;\vdots\; -\boldsymbol{\mu}_*'T(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*) \;\vdots\; \sqrt{T}(\ddot{\nu}_T - \nu_*) \;]' + o_{\mathbb{P}}(1) \Rightarrow [\; \boldsymbol{\mathscr{L}}_1' \;\vdots\; -\boldsymbol{\mu}_*'\boldsymbol{\mathscr{L}}_1 \;\vdots\; \mathscr{L}_3 \;]'$$

by Lemma 3 (*i*). ■

Proof of Lemma 4. (*i*) We first note that $\ddot{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) = (\ddot{\mathbf{D}}^{-1}\sum_{t=1}^T \ddot{\boldsymbol{z}}_t\ddot{\boldsymbol{z}}_t'\ddot{\mathbf{D}}^{-1})^{-1}\ddot{\mathbf{D}}^{-1}\sum_{t=1}^T \ddot{\boldsymbol{z}}_t e_t$ from (14), and Lemma 2 (*vi*) implies that $\ddot{\mathbf{D}}^{-1}(\sum_{t=1}^T \ddot{\boldsymbol{z}}_t\ddot{\boldsymbol{z}}_t')\ddot{\mathbf{D}}^{-1} \Rightarrow \boldsymbol{\mathscr{M}}$. We therefore focus on the limit distribution of $\ddot{\mathbf{D}}^{-1}\sum_{t=1}^T \ddot{\boldsymbol{z}}_t e_t$. We note that $\ddot{\mathbf{D}}^{-1}\sum_{t=1}^T \ddot{\boldsymbol{z}}_t e_t = \ddot{\mathbf{D}}^{-1}\sum_{t=1}^T [\ddot{u}_{t-1}e_r, \ddot{\mathbf{m}}_{t-1}'e_t, (t-1)e_t, e_t, \boldsymbol{z}_{2t}'e_t]'$.

We now investigate the asymptotic behavior of each element on the right side. First, we already showed in the proof of Lemma B.6 (*iv*) that $\ddot{u}_t = u_t + O_{\mathbb{P}}(T^{-1/2})$. Therefore, $T^{-1/2}\sum_{t=1}^T \ddot{u}_{t-1}e_t = T^{-1/2}\sum_{t=1}^T u_{t-1}e_t + o_{\mathbb{P}}(1)$. In addition, $T^{-1/2}\sum_{t=1}^T u_{t-1}e_t = T^{-1}\sum_{t=1}^T \sqrt{T}(B_{ueT}(t/T) - B_{ueT}((t-1)/T)) = \int dB_{ueT}(r) \Rightarrow \mathscr{J}_1 := \int d\mathscr{B}_{ue}$. Second, we note that $\ddot{\mathbf{m}}_{t-1} = \mathbf{m}_{t-1} - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)(t-1)$. Therefore, $T^{-1}\sum_{t=1}^T \ddot{\mathbf{m}}_{t-1}e_t = T^{-1}\sum_{t=1}^T \mathbf{m}_{t-1}e_t - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)T^{-1}\sum_{t=1}^T (t-1)e_t$. We here note that $T^{-1}\sum_{t=1}^T \mathbf{m}_{t-1}e_t \Rightarrow \int \boldsymbol{\mathscr{B}}_s d\mathscr{B}_e$ by Lemma B.7 (*iv*) and $\sqrt{T}(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*) \Rightarrow 3\int r\boldsymbol{\mathscr{B}}_s$ by Lemma B.7 (*i*). In addition to this, $T^{-3/2}\sum_{t=1}^T (t-1)e_t = T^{-1}\sum_{t=1}^T \frac{(t-1)}{T}\sqrt{T}(B_{eT}((t-1)/T) - B_{eT}((t-2)/T)) = \int r dB_{eT}(r) \Rightarrow \int r d\mathscr{B}_e$. Hence, it follows that $T^{-1}\sum_{t=1}^T \ddot{\mathbf{m}}_{t-1}e_t \Rightarrow \boldsymbol{\mathscr{J}}_2 := \int \boldsymbol{\mathscr{B}}_s d\mathscr{B}_e - 3\int r\boldsymbol{\mathscr{B}}_s \int r d\mathscr{B}_u = \int \bar{\boldsymbol{\mathscr{B}}}_s d\mathscr{B}_e$. Third, it is already showed that $T^{-3/2}\sum_{t=1}^T (t-1)e_{t-1} \Rightarrow \mathscr{J}_3 := \int r d\mathscr{B}_e$. Fourth, we note that $T^{-1/2}\sum_{t=1}^T e_t = T^{-1}\sum_{t=1}^T \sqrt{T}(B_{eT}(t/T) - B_{eT}((t-1)/T)) = \int dB_{eT}(r) \Rightarrow \mathscr{J}_4 := \int d\mathscr{B}_e$. Fifth, note that $T^{-1/2}\sum_{t=1}^T \boldsymbol{z}_{2t}e_t = T^{-1}\sum_{t=1}^T \sqrt{T}(\mathbf{B}_{zeT}(t/T) - \mathbf{B}_{zeT}((t-1)/T)) = \int d\mathbf{B}_{zeT}(r) \Rightarrow \boldsymbol{\mathscr{J}}_5 := \int d\boldsymbol{\mathscr{B}}_{ze}$. We next combine the first to fifth facts to obtain that $\ddot{\mathbf{D}}^{-1}\sum_{t=1}^T \ddot{\boldsymbol{z}}_t e_t \Rightarrow \boldsymbol{\mathscr{J}}$

by noting that

$$\boldsymbol{\mathscr{J}} = \left[\begin{array}{ccccc} \mathscr{J}_1 & \boldsymbol{\mathscr{J}}_2' & \mathscr{J}_3 & \mathscr{J}_4 & \boldsymbol{\mathscr{J}}_5' \end{array}\right]' := \left[\begin{array}{ccccc} \int d\mathscr{B}_{ue} & \int \bar{\boldsymbol{\mathscr{B}}}_s' d\mathscr{B}_e & \int r d\mathscr{B}_e & \int d\mathscr{B}_e & \int d\boldsymbol{\mathscr{B}}_{\boldsymbol{z}e}' \end{array}\right]',$$

leading to that $\ddot{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) \Rightarrow \boldsymbol{\mathscr{M}}^{-1}\boldsymbol{\mathscr{J}}$, as desired.

(*ii*) From the definitions of $\boldsymbol{\tau}_{T*} := [\rho_*, \boldsymbol{\tau}_{1T}', \boldsymbol{\kappa}_*']'$ and $\boldsymbol{\tau}_{1T} := [(\boldsymbol{\eta}_* + \rho_*(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*))', \varsigma_* + \boldsymbol{\eta}_*'(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*) + \rho_*(\ddot{\vartheta}_T - \vartheta_{T*}), \gamma_* + \rho_*(\ddot{\nu}_T - \nu_*)]'$, we obtain that $\ddot{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) = \ddot{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_*) + [0, -\rho_* T(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*)', -\rho_* T^{3/2}(\ddot{\vartheta}_T - \vartheta_{T*}), -\rho_* T^{1/2}(\ddot{\nu}_T - \nu_*), \mathbf{0}']' \Rightarrow \boldsymbol{\mathscr{D}}$ by Lemma 4 (*i*) and noting that $\boldsymbol{\eta}_* = \mathbf{0}$. In addition, Lemma 3 implies that $T(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*) \Rightarrow \boldsymbol{\mathscr{L}}_1'$, and $T^{1/2}(\ddot{\nu}_T - \nu_*) \Rightarrow \mathscr{L}_3$. Furthermore, the proof of Lemma 3 (*i*) shows that $T^{3/2}(\ddot{\vartheta}_T - \vartheta_{T*}) \Rightarrow \mathscr{L}_2$. Therefore, $\ddot{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_*) \Rightarrow \boldsymbol{\mathscr{D}} - \rho_*[0, \boldsymbol{\mathscr{L}}_1', \mathscr{L}_2, \mathscr{L}_3, \mathbf{0}']' = \boldsymbol{\mathscr{D}} - \rho_*[0, \boldsymbol{\mathscr{L}}', \mathbf{0}']'$. This completes the proof. ■

Proof of Theorem 1. (*i*) We note that $(\widehat{\boldsymbol{\alpha}}_T - \boldsymbol{\alpha}_*) = (\mathbf{T}_T - \mathbf{T})\boldsymbol{\tau}_* + \mathbf{T}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_*) + (\mathbf{T}_T - \mathbf{T})(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_*) = (\mathbf{T}_T - \mathbf{T})\boldsymbol{\tau}_* + \mathbf{T}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_*) + o_{\mathbb{P}}(\mathbf{T}_T - \mathbf{T})$ using (10) and (11). We further note that

$$
\begin{aligned}
&(\mathbf{T}_T - \mathbf{T})\boldsymbol{\tau}_* + \mathbf{T}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_*) \\
&\qquad = \begin{bmatrix} (\ddot{\rho}_T - \rho_*) \\ -\boldsymbol{\beta}_*(\ddot{\rho}_T - \rho_*) + (\ddot{\boldsymbol{\eta}}_T - \boldsymbol{\eta}_*) - \rho_*(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*) \\ (\ddot{\varsigma}_T - \varsigma_*) - \boldsymbol{\mu}_*'(\ddot{\boldsymbol{\eta}}_T - \boldsymbol{\eta}_*) - \zeta_*(\ddot{\rho}_T - \rho_*) - \rho_*(\widetilde{\zeta}_T - \zeta_*) - \boldsymbol{\eta}_*'(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*) \\ (\ddot{\gamma}_T - \gamma_*) - \rho_*(\widetilde{\nu}_T - \nu_*) - \nu_*(\ddot{\rho}_T - \rho_*) \\ (\ddot{\boldsymbol{\kappa}}_T - \boldsymbol{\kappa}_*) \end{bmatrix}
\end{aligned}
$$

and the fact that $\widetilde{\boldsymbol{\beta}}_T = \ddot{\boldsymbol{\beta}}_T$, $\widetilde{\zeta}_T = -\ddot{\boldsymbol{\mu}}_T'\ddot{\boldsymbol{\beta}}_T + \ddot{\vartheta}_T$ and $\zeta_* = -\boldsymbol{\mu}_*'\boldsymbol{\beta}_* + \vartheta_*$, so that $(\widetilde{\zeta}_T - \zeta_*) = (\ddot{\vartheta}_T - \vartheta_{T*}) - \boldsymbol{\mu}_*'(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*) - (\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)'(\widetilde{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*)$, where $(\ddot{\vartheta}_T - \vartheta_{T*}) = O_{\mathbb{P}}(T^{-3/2})$ and $(\ddot{\boldsymbol{\mu}}_T - \boldsymbol{\mu}_*)'(\widetilde{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*) = O_{\mathbb{P}}(T^{-3/2})$ by Lemmas 4 (*i*), B.7 (*i*), and 3 (*ii*). Therefore, it now follows that

$$
\begin{aligned}
&(\mathbf{T}_T - \mathbf{T})\boldsymbol{\tau}_* + \mathbf{T}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_*) \\
&= (\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) - (\ddot{\rho}_T - \rho_*) \begin{bmatrix} 0 \\ \boldsymbol{\beta}_* \\ \zeta_* \\ \nu_* \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{0} \\ -\boldsymbol{\mu}_*'\{(\ddot{\boldsymbol{\eta}}_T - \boldsymbol{\eta}_*) - \rho_*(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*)\} + O_{\mathbb{P}}(T^{-3/2}) \\ 0 \\ \mathbf{0} \end{bmatrix}. \qquad \text{(A.4)}
\end{aligned}
$$

We here use the fact that $\boldsymbol{\eta}_* = \mathbf{0}$. We further note that $\sqrt{T}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) \Rightarrow [\mathscr{D}_1, \mathbf{0}_{2k\times 1}', 0, \mathscr{D}_4, \boldsymbol{\mathscr{D}}_5]'$ by

Lemma 4 (i) and $-\sqrt{T}(\ddot{\rho}_T - \rho_*)\left[0, \boldsymbol{\beta}_*', \zeta_*, \nu_*, \mathbf{0}'\right] \Rightarrow -\mathscr{D}_1\left[0, \boldsymbol{\beta}_*', \zeta_*, \nu_*, \mathbf{0}'\right]$. In addition, we note that $-\boldsymbol{\mu}_*'\sqrt{T}\{(\ddot{\boldsymbol{\eta}}_T - \boldsymbol{\eta}_*) - \rho_*(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*)\} + O_{\mathbb{P}}(T^{-1}) = o_{\mathbb{P}}(1)$ because $\sqrt{T}\{(\ddot{\boldsymbol{\eta}}_T - \boldsymbol{\eta}_*) - \rho_*(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*)\} = O_{\mathbb{P}}(T^{-1/2})$ by Lemma 4 (i). Therefore, it follows that

$$\sqrt{T}(\widehat{\boldsymbol{\alpha}}_T - \boldsymbol{\alpha}_*) \Rightarrow [\mathscr{D}_1, -\boldsymbol{\beta}_*'\mathscr{D}_1, -\zeta_*\mathscr{D}_1, \mathscr{D}_4 - \nu_*\mathscr{D}_1, \boldsymbol{\mathscr{D}}_5']'. \tag{A.5}$$

We finally note that the derived weak limit is identical to $\boldsymbol{c}_*\mathscr{D}_1 + [0, \mathbf{0}', 0, 0, \mathscr{D}_4, \boldsymbol{\mathscr{D}}_5']'$.

(ii, iii, and iv) For an efficient proof, we prove (ii, iii, and iv) together by supposing that $\boldsymbol{\beta}_* = \mathbf{0}$ and $\zeta_* = 0$, although it is not permitted by $\rho_* < 0$. If so, it algebraically follows from (A.4) that

$$\begin{aligned}
(\widehat{\boldsymbol{\alpha}}_T - \boldsymbol{\alpha}_*) &= (\mathbf{T}_T - \mathbf{T})\boldsymbol{\tau}_* + \mathbf{T}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_*) + o_{\mathbb{P}}(\mathbf{T}_T - \mathbf{T}) \\
&= (\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) + \begin{bmatrix} 0 \\ \mathbf{0} \\ -\boldsymbol{\mu}_*'\{(\ddot{\boldsymbol{\eta}}_T - \boldsymbol{\eta}_*) - \rho_*(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*)\} + O_{\mathbb{P}}(T^{-3/2}) \\ -\nu_*(\ddot{\rho}_T - \rho_*) \\ \mathbf{0} \end{bmatrix} + o_{\mathbb{P}}(\mathbf{T}_T - \mathbf{T}).
\end{aligned}$$

Here, $\widehat{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) \Rightarrow [\mathscr{D}_1, \boldsymbol{\mathscr{D}}_2', 0, \mathscr{D}_4, \boldsymbol{\mathscr{D}}_5]'$ and $T\{(\ddot{\boldsymbol{\eta}}_T - \boldsymbol{\eta}_*) - \rho_*(\ddot{\boldsymbol{\beta}}_T - \boldsymbol{\beta}_*)\} \Rightarrow \boldsymbol{\mathscr{D}}_2$ by Lemma 4 (i), so that

$$\widehat{\mathbf{D}}(\widehat{\boldsymbol{\alpha}}_T - \boldsymbol{\alpha}_*) \Rightarrow [\mathscr{D}_1, \boldsymbol{\mathscr{D}}_2', -\boldsymbol{\mu}_*'\boldsymbol{\mathscr{D}}_2, \mathscr{D}_4 - \nu_*\mathscr{D}_1, \boldsymbol{\mathscr{D}}_5']', \tag{A.6}$$

where $\widehat{\mathbf{D}} := \mathrm{diag}[\sqrt{T}, T\mathbf{I}_{2k+1}, \sqrt{T}, \ddot{\mathbf{D}}_2]$. From the weak limits in (A.5) and (A.6), (ii, iii, and iv) follow by applying the weak limits corresponding to the conditions in (ii, iii, and iv). For example, if the conditions in (ii) are imposed, the weak limit of $\widehat{\boldsymbol{\theta}}_T^+$ is determined by (A.6) as $\boldsymbol{\mathscr{D}}_2^+$, but the others are determined by (A.5), so that the weak limit in (ii) is delivered. As another example, if the conditions in (iv) are imposed, the weak limit of $\widehat{\xi}_T$ is determined by (A.6) as $\boldsymbol{\mu}_*'\boldsymbol{\mathscr{D}}_2$, but the others are determined by (A.5), so that the weak limit in (iv) is delivered. In this manner, the weak limits in (ii, iii, and iv) are obtained. ■

Proof of Theorem 2. (i) We show the null limit distribution of each test using Lemma 4.

($i.a$) Note that $(\widehat{\boldsymbol{\theta}}_T^+ - \widehat{\boldsymbol{\theta}}_T^-) = (\boldsymbol{\theta}_*^+ - \boldsymbol{\theta}_*^-) - (\ddot{\rho}_T - \rho_*)(\boldsymbol{\beta}_*^+ - \boldsymbol{\beta}_*^-) + (\ddot{\boldsymbol{\eta}}_T^+ - \boldsymbol{\eta}_{T*}^+) - (\ddot{\boldsymbol{\eta}}_T^- - \boldsymbol{\eta}_{T*}^-) + o_{\mathbb{P}}(T^{-1})$, so that it follows that $(\widehat{\boldsymbol{\theta}}_T^+ - \widehat{\boldsymbol{\theta}}_T^-) = (\ddot{\boldsymbol{\eta}}_T^+ - \boldsymbol{\eta}_{T*}^+) - (\ddot{\boldsymbol{\eta}}_T^- - \boldsymbol{\eta}_{T*}^-) + o_{\mathbb{P}}(T^{-1})$ under $\mathscr{H}_0'$. Therefore, $\widehat{\mathbf{R}}_1\widehat{\boldsymbol{\alpha}}_T =$

$\mathbf{R}_1(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) + o_{\mathbb{P}}(T^{-1})$, and it follows from Lemmas 4 (*i*) and B.7 (*v*) that

$$W_T^{(1)} = (\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*})'\ddot{\mathbf{D}}\mathbf{R}_1' \left(\widehat{\sigma}_{e,T}^2 \mathbf{R}_1 \left(\ddot{\mathbf{D}}^{-1}\sum_{t=1}^T \ddot{\boldsymbol{z}}_t\ddot{\boldsymbol{z}}_t'\ddot{\mathbf{D}}^{-1}\right)^{-1} \mathbf{R}_1'\right)^{-1} \mathbf{R}_1\ddot{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) + o_{\mathbb{P}}(1)$$
$$\Rightarrow \boldsymbol{\mathscr{D}}'\mathbf{R}_1'\left(\sigma_e^2\mathbf{R}_1\boldsymbol{\mathscr{M}}^{-1}\mathbf{R}_1'\right)^{-1}\mathbf{R}_1\boldsymbol{\mathscr{D}}.$$

(*i.b*) We note that $(\widehat{\boldsymbol{\pi}}_T^+, \widehat{\boldsymbol{\pi}}_T^-) = (\ddot{\boldsymbol{\pi}}_T^+, \ddot{\boldsymbol{\pi}}_T^-)$. Therefore, it follows from Lemmas 4 (*i*) and B.7 (*v*) that

$$W_T^{(2)} = (\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*})'\ddot{\mathbf{D}}\widehat{\mathbf{R}}_2' \left(\widehat{\sigma}_{e,T}^2 \widehat{\mathbf{R}}_2 \left(\ddot{\mathbf{D}}^{-1}\sum_{t=1}^T \ddot{\boldsymbol{z}}_t\ddot{\boldsymbol{z}}_t'\ddot{\mathbf{D}}^{-1}\right)^{-1} \widehat{\mathbf{R}}_2'\right)^{-1} \widehat{\mathbf{R}}_2\ddot{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) + o_{\mathbb{P}}(1)$$
$$\Rightarrow \boldsymbol{\mathscr{D}}'\widehat{\mathbf{R}}_2'\left(\sigma_e^2\widehat{\mathbf{R}}_2\boldsymbol{\mathscr{M}}^{-1}\widehat{\mathbf{R}}_2'\right)^{-1}\widehat{\mathbf{R}}_2\boldsymbol{\mathscr{D}}.$$

(*i.c*) From the notice given in the proofs of (*i.a* and *i.b*), it follows from Lemmas 4 (*i*) and B.7 (*v*) that

$$W_T^{(3)} = (\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*})'\ddot{\mathbf{D}}\mathbf{R}_3' \left(\widehat{\sigma}_{e,T}^2 \mathbf{R}_3 \left(\ddot{\mathbf{D}}^{-1}\sum_{t=1}^T \ddot{\boldsymbol{z}}_t\ddot{\boldsymbol{z}}_t'\ddot{\mathbf{D}}^{-1}\right)^{-1} \mathbf{R}_3'\right)^{-1} \mathbf{R}_3\ddot{\mathbf{D}}(\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) + o_{\mathbb{P}}(1)$$
$$\Rightarrow \boldsymbol{\mathscr{D}}'\mathbf{R}_3'\left(\sigma_e^2\mathbf{R}_3\boldsymbol{\mathscr{M}}^{-1}\mathbf{R}_3'\right)^{-1}\mathbf{R}_3\boldsymbol{\mathscr{D}}.$$

(*ii*) We show the power behavior of each test using Lemma 4.

(*ii.a*) We show that $W_T^{(1)} = O_{\mathbb{P}}(T)$. Note that $(\widehat{\boldsymbol{\theta}}_T^+ - \widehat{\boldsymbol{\theta}}_T^-) = (\boldsymbol{\theta}_*^+ - \boldsymbol{\theta}_*^-) - (\ddot{\rho}_T - \rho_*)(\boldsymbol{\beta}_*^+ - \boldsymbol{\beta}_*^-) + o_{\mathbb{P}}(T^{-1/2})$ under $\mathscr{H}_1'$ and $(\ddot{\rho}_T - \rho_*) = O_{\mathbb{P}}(T^{-1/2})$, so that $(\widehat{\boldsymbol{\theta}}_T^+ - \widehat{\boldsymbol{\theta}}_T^-) - (\boldsymbol{\theta}_*^+ - \boldsymbol{\theta}_*^-) = O_{\mathbb{P}}(T^{-1/2})$. Therefore, $(\widehat{\boldsymbol{\theta}}_T^+ - \widehat{\boldsymbol{\theta}}_T^-) = O_{\mathbb{P}}(T^{1/2})$, and this implies that $W_T^{(1)} = O_{\mathbb{P}}(T)$.

(*ii.b*) We show that $W_T^{(2)} = O_{\mathbb{P}}(T)$. Note that Lemma 4 (*i*) implies that $\ddot{\boldsymbol{\pi}}_T^+ - \ddot{\boldsymbol{\pi}}_T^- = \boldsymbol{\pi}_*^+ - \boldsymbol{\pi}_*^- + O_{\mathbb{P}}(T^{-1/2})$. Therefore, $\ddot{\boldsymbol{\pi}}_T^+ - \ddot{\boldsymbol{\pi}}_T^- = O_{\mathbb{P}}(T^{1/2})$, implying that $W_T^{(2)} = O_{\mathbb{P}}(T)$.

(*ii.c*) From the proofs of (*ii.a*) and (*ii.b*), $(\widehat{\boldsymbol{\theta}}_T^+ - \widehat{\boldsymbol{\theta}}_T^-) = O_{\mathbb{P}}(T^{1/2})$ and $(\ddot{\boldsymbol{\pi}}_T^+ - \ddot{\boldsymbol{\pi}}_T^-) = O_{\mathbb{P}}(T^{1/2})$. Therefore, it trivially follows that $W_T^{(3)} = O_{\mathbb{P}}(T)$.

(*iii*) We show the null limit distribution of each test using Lemmas 3 and 4.

(*iii.a*) First, Lemma B.6 (*iv*) implies that $\ddot{\sigma}_{u,T}^2 \overset{\mathbb{P}}{\to} \sigma_u^2$, and $\ddot{\mathbf{D}}_1^{-1}\sum_{t=1}^T \ddot{\boldsymbol{r}}_{t-1}\ddot{\boldsymbol{r}}_{t-1}'\ddot{\mathbf{D}}_1^{-1} \Rightarrow \boldsymbol{\mathscr{M}}_{11}$ from Lemma 2 (*vii*). Therefore, $\ddot{\mathbf{W}}_T^{(1)} \Rightarrow \sigma_u^2\ddot{\mathbf{R}}_1\boldsymbol{\mathscr{M}}_{11}^{-1}\ddot{\mathbf{R}}_1'$. Second, $\mathbb{H}_0'$ implies that $\ddot{\mathbf{R}}_1\ddot{\mathbf{D}}_1\ddot{\boldsymbol{v}}_T = T(\ddot{\boldsymbol{\beta}}_T^+ - \boldsymbol{\beta}_*^+) - T(\ddot{\boldsymbol{\beta}}_T^- - \boldsymbol{\beta}_*^-) \Rightarrow (\boldsymbol{\mathscr{L}}_1^+ - \boldsymbol{\mathscr{L}}_1^-)$. If we combine these two facts, it follows that $\mathscr{W}_T^{(1)} \Rightarrow (\boldsymbol{\mathscr{L}}_1^+ - \boldsymbol{\mathscr{L}}_1^-)'(\sigma_u^2\ddot{\mathbf{R}}_1\boldsymbol{\mathscr{M}}_{11}^{-1}\ddot{\mathbf{R}}_1')^{-1}(\boldsymbol{\mathscr{L}}_1^+ - \boldsymbol{\mathscr{L}}_1^-)$ that is identical to the given weak limit by the definitions of $\ddot{\mathbf{R}}_1$ and $\boldsymbol{\mathscr{L}}$.

(*iii.b*) First, Lemma B.7 (*v*) implies that $\widehat{\sigma}_{e,T}^2 \overset{\mathbb{P}}{\to} \sigma_e^2$, and $\ddot{\mathbf{D}}^{-1}\sum_{t=1}^T \ddot{\boldsymbol{z}}_t\ddot{\boldsymbol{z}}_t'\ddot{\mathbf{D}}^{-1} \Rightarrow \boldsymbol{\mathscr{M}}$ from Lemma 2

(vi). Therefore, $\ddot{\mathbf{W}}_T^{(2)} \Rightarrow \sigma_e^2 \widehat{\mathbf{R}}_2 \boldsymbol{\mathscr{M}}^{-1} \widehat{\mathbf{R}}_2'$. Second, $\widehat{\mathbf{R}}_2 \ddot{\mathbf{D}} \ddot{\boldsymbol{\tau}}_T = \sqrt{T}(\ddot{\boldsymbol{\pi}}_T^+ - \ddot{\boldsymbol{\pi}}_T^-) \Rightarrow \boldsymbol{\mathscr{D}}_5^+ - \boldsymbol{\mathscr{D}}_5^- = \widehat{\mathbf{R}}_2 \boldsymbol{\mathscr{D}}$. If we combine these two facts, it follows that $\mathscr{W}_T^{(2)} \Rightarrow \boldsymbol{\mathscr{D}}' \widehat{\mathbf{R}}_2' (\sigma_2^2 \widehat{\mathbf{R}}_2 \boldsymbol{\mathscr{M}}^{-1} \widehat{\mathbf{R}}_2')^{-1} \widehat{\mathbf{R}}_2 \boldsymbol{\mathscr{D}}$.

($iii.c$) We note that from the definition of $\mathscr{W}_T^{(3)}$, it follows that $\mathscr{W}_T^{(3)} = \mathscr{W}_T^{(1)} + \mathscr{W}_T^{(2)}$, and it follows from ($i.a$ and $i.b$) that $\mathscr{W}_T^{(3)} \Rightarrow \boldsymbol{\mathscr{L}}' \ddot{\mathbf{R}}_1' (\sigma_u^2 \ddot{\mathbf{R}}_1 \boldsymbol{\mathscr{M}}_{11}^{-1} \ddot{\mathbf{R}}_1')^{-1} \ddot{\mathbf{R}}_1 \boldsymbol{\mathscr{L}} + \boldsymbol{\mathscr{D}}' \widehat{\mathbf{R}}_2' (\sigma_2^2 \widehat{\mathbf{R}}_2 \boldsymbol{\mathscr{M}}^{-1} \widehat{\mathbf{R}}_2')^{-1} \widehat{\mathbf{R}}_2 \boldsymbol{\mathscr{D}}$ under $\mathbb{H}_0'''$.

(iv) We next show the power behavior of each test.

($iv.a$) To show the claim, we show that $\mathscr{W}_T^{(1)} = O_{\mathbb{P}}(T^2)$ under $\mathbb{H}_1'$. Given that $\ddot{\mathbf{W}}_T^{(1)} \Rightarrow \sigma_u^2 \ddot{\mathbf{R}}_1 \boldsymbol{\mathscr{M}}_{11}^{-1} \ddot{\mathbf{R}}_1'$, we focus on the limit behavior of $\ddot{\mathbf{R}}_1 \ddot{\mathbf{D}}_1 \ddot{\boldsymbol{\upsilon}}_T$. Note that $\ddot{\mathbf{R}}_1 \ddot{\mathbf{D}}_1 (\ddot{\boldsymbol{\upsilon}}_T - \bar{\boldsymbol{\upsilon}}_{T*}) \Rightarrow (\boldsymbol{\mathscr{L}}_1^+ - \boldsymbol{\mathscr{L}}_1^-)$ by Lemma 3 (i) and $\ddot{\mathbf{R}}_1 \ddot{\mathbf{D}}_1 \bar{\boldsymbol{\upsilon}}_{T*} = T\boldsymbol{\beta}_* = O(T)$. Therefore, $\ddot{\mathbf{R}}_1 \ddot{\mathbf{D}}_1 \ddot{\boldsymbol{\upsilon}}_T = O_{\mathbb{P}}(T)$, implying that $\mathscr{W}_T^{(1)} = O_{\mathbb{P}}(T^2)$, leading to the desired result.

($iv.b$) We show that $\mathscr{W}_T^{(2)} = O_{\mathbb{P}}(T)$ under $\mathscr{H}_1''$. Given that $\ddot{\mathbf{W}}_T^{(2)} \Rightarrow \sigma_e^2 \widehat{\mathbf{R}}_2 \boldsymbol{\mathscr{M}}^{-1} \widehat{\mathbf{R}}_2'$, we focus on the limit behavior of $\widehat{\mathbf{R}}_2 \ddot{\mathbf{D}} \ddot{\boldsymbol{\tau}}_T$. We note that $\widehat{\mathbf{R}}_2 \ddot{\mathbf{D}} (\ddot{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{T*}) = \sqrt{T}[(\ddot{\boldsymbol{\pi}}_T^+ - \ddot{\boldsymbol{\pi}}_T^-) - (\boldsymbol{\pi}_*^+ - \boldsymbol{\pi}_*^-)] \Rightarrow \widehat{\mathbf{R}}_2 \boldsymbol{\mathscr{D}}$ by Lemma 4 (i) and $\widehat{\mathbf{R}}_2 \ddot{\mathbf{D}} \boldsymbol{\tau}_{T*} = \sqrt{T}(\boldsymbol{\pi}_*^+ - \boldsymbol{\pi}_*^-) = O(T^{1/2})$. Therefore, $\widehat{\mathbf{R}}_2 \ddot{\mathbf{D}} \ddot{\boldsymbol{\tau}}_T = O_{\mathbb{P}}(T^{1/2})$, implying that $\mathscr{W}_T^{(2)} = O_{\mathbb{P}}(T)$.

($iv.c$) We show that $\mathscr{W}_T^{(3)} = O_{\mathbb{P}}(T^2)$ under $\mathbb{H}_1'''$. Note that $\mathscr{W}_T^{(3)} = \mathscr{W}_T^{(1)} + \mathscr{W}_T^{(2)}$, and $\mathscr{W}_T^{(1)}$ and $\mathscr{W}_T^{(2)}$ are $O_{\mathbb{P}}(T^2)$ and $O_{\mathbb{P}}(T)$ by ($ii.a$) and ($ii.b$), respectively. Therefore, $\mathscr{W}_T^{(3)} = O_{\mathbb{P}}(T^2)$ under $\mathbb{H}_{11}'''$ and $\mathscr{W}_T^{(3)} = O_{\mathbb{P}}(T)$ under $\mathbb{H}_{01}''' \bigcap \mathbb{H}_{12}'''$. This completes the proof. ■

A.4 Additional Monte Carlo Simulations

In this section, we compare the finite sample performance of OLS with 2SNARDL and also provide additional simulation evidence for the main claims.

A.4.1 Comparison of OLS with 2SNARDL

In this section, we compare the finite sample performance of OLS with 2SNARDL by simulation.

For the simulation, we assume the following simulation environment. We first suppose that

$$\Delta y_t = \rho_* u_{t-1} + \varphi_* \Delta y_{t-1} + \pi_*^+ \Delta x_t^+ + \pi_*^- \Delta x_t^- + e_t,$$

$$u_t = y_t - \beta_*^+ x_t^+ - \beta_*^- x_t^- - \zeta_*(t-1), \quad \text{and} \quad \Delta x_t = \omega_* + \varrho_* \Delta x_{t-1} + v_t,$$

where $(e_t, v_t)' \sim$ IID $N(\mathbf{0}_2, \mathbf{I}_2)$ and $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 3/2, -3/2, 0)$. We further assume two DGP conditions by letting (i) $(\omega_*, \varrho_*) = (1/2, 0)$ or (ii) $(\omega_*, \varrho_*) = (1/4, 1/2)$. That is, we let $\{\Delta x_t\}$ be an independent or autocorrelated series. For each DGP, we plug the equation for u_t into the equation for Δy_t to obtain

$$\Delta y_t = \alpha_* + \rho_* y_{t-1} + \theta_*^+ x_{t-1}^+ + \theta_*^- x_{t-1}^- + \xi_*(t-1) + \varphi_* \Delta y_{t-1} + \pi_*^+ \Delta x_t^+ + \pi_*^- \Delta x_t^- + e_t$$

and estimate the coefficients in this equation by OLS and 2SNARDL. For the 2SNARDL, we first estimate ρ_*, β_*^+, and β_*^- by following Cho, Greenwood-Nimmo, and Shin (2023a) and next indirectly estimate θ_*^+ and θ_*^- by noting that $\theta_*^+ = -\rho_* \beta_*^+$ and $\theta_*^- = -\rho_* \beta_*^-$.

Using the OLS and 2SNARDL estimates, we report their finite sample biases and root mean square errors (RMSEs). For example, the finite sample bias and RMSE of $\widehat{\theta}_T^+$ are computed as follows:

$$\text{Bias} = \frac{1}{m} \sum_{i=1}^{m} (\widehat{\theta}_{T,j}^+ - \theta_*^+) \quad \text{and} \quad \text{RMSE} = \sqrt{\frac{1}{m} \sum_{i=1}^{m} (\widehat{\theta}_{T,j}^+ - \theta_*^+)^2},$$

where j indicates the experimental index, and m denotes the total number of experiments. For the simulation, we let $m = 50000$.

Table A.1 reports the finite sample biases of the OLS and 2SNARDL estimators when $\{\Delta x_t\}$ is an independent series. As we see from the table, the performance of the OLS estimator is comparable to the 2SNARDL estimator. As T increases, the finite sample bias converges to zero for both estimators, which implies that the asymptotic bias is negligible. Nonetheless, each parameter estimator has nuanced results. Specifically, when T is small, say $T = 100$, the finite sample biases of the OLS estimator are greater than those of the 2SNARDL estimator for the long-run parameters. If the sample size is as large as 2000, this relationship is reversed. For the short-run parameter, the OLS estimator for ρ_* always exhibits bigger biases than 2SNARDL, but this relationship is reversed for π_*^+ and π_*^-. For the case of φ_*, both estimators exhibit similar finite sample biases.

Table A.2 reports the finite sample RMSEs of the OLS and 2SNARDL estimators when $\{\Delta x_t\}$ is an independent series. From the table, as T increases, the finite sample RMSE converges to zero for both estimators, which implies that both estimators are consistent for the unknown parameters. We further observe that both estimators exhibit similar RMSEs for each sample size. It is difficult to say one estimator is superior to another in terms of finite sample RMSE. This aspect implies that the 2SNARDL estimator for θ_*^+ and θ_*^- has the same convergence rate as for the OLS estimator, although the 2SNARDL estimators for β_*^+ and β_*^- are super-consistent. The main reason of this is in the fact that the 2SNARDL estimator for ρ_* has $\sqrt{T}$ convergence rate, so that the convergence rate of the 2SNARDL estimator for θ_*^+ and θ_*^- is determined by the 2SNARDL estimator for ρ_*.

Tables A.3 and A.4 report the finite sample biases and RMSEs of the OLS and 2SNARDL estimators, respectively, when $\{\Delta x_t\}$ is an autocorrelated series. Although there exist minor differences between the tables, their qualitative properties are identical to those of Tables A.1 and A.2, respectively. This aspect implies that the simulation results are the same irrespective of whether Δx_t is serially correlated or not.

From this experiment, we observe that both OLS and 2SNARDL estimators are consistent estimators. As T increases, the finite sample biases and RMSEs decrease. Although the performances of the two estimators in terms of finite sample bias depend on multiple factors such as the sample size and the roles of the parameters, the overall performance of the OLS estimator is not so different from 2SNARDL in terms of their finite sample RMSEs.

A.4.2 Additional Simulation Evidence

In this section, we provide additional simulation evidence for the main claims. The main goal of this section is to verify the properties in Theorem 1.

Theorem 1 demonstrates that the convergence rate of the OLS estimator depends on the DGP conditions. Specifically, four different DGP conditions are considered in Theorem 1 as follows:

- Theorem 1 (*i*): for each $j = 1, 2, \ldots, k$, $\beta_{j*}^{\pm} \neq 0$ and $\zeta_* \neq 0$;
- Theorem 1 (*ii*): $\boldsymbol{\beta}_*^{+} = \mathbf{0}$, but for each $j = 1, 2, \ldots, k$, $\beta_{j*}^{-} \neq 0$, and $\zeta_* \neq 0$;
- Theorem 1 (*iii*): $\boldsymbol{\beta}_*^{-} = \mathbf{0}$, but for each $j = 1, 2, \ldots, k$, $\beta_{j*}^{+} \neq 0$, and $\zeta_* \neq 0$; and
- Theorem 1 (*iv*): for each $j = 1, 2, \ldots, k$, $\beta_{j*}^{\pm} \neq 0$ but $\zeta_* = 0$,

and Theorem 1 shows that the convergence rate of the OLS estimator is different under each condition. We verify this property by simulation.

DGP Condition under Theorem 1 (*i*) For simulation, we first generate data according to the DGP condition in Theorem 1 (*i*). Specifically, we generate data observations according to the DGP condition in Section A.4.1 by letting $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 3/2, -3/2, 1)$ and $(\omega_*, \varrho_*) = (1/2, 0)$. Using the data set generated by this DGP condition, we compute OLS estimators by repeating independent experiments 50000 times.

We corroborate Theorem 1 (*i*) in two ways. First, we compute the finite sample bias and RMSE of the OLS estimator. For this, we compute the bias and RMSE by multiplying the convergence rate in Theorem 1 (*i*) to the OLS estimator. For example, as $\widehat{\theta}_T^+$ has convergence rate $\sqrt{T}$, we compute the bias and RMSE as

$$\text{Bias} = \frac{1}{m}\sum_{j=1}^{m}\sqrt{T}(\widehat{\theta}_{T,j}^{+} - \theta_*^{+}) \quad \text{and} \quad \text{RMSE} = \sqrt{\frac{1}{m}\sum_{j=1}^{m}T(\widehat{\theta}_{T,j}^{+} - \theta_*^{+})^2},$$

where m denotes the total number of experiments, viz., 50000. If the convergence rate in Theorem 1 (*i*) is correct for the OLS estimator, the finite sample bias and RMSE have to converge to zero and a positive constant as the sample size increases. Otherwise, the finite sample RMSE would converge to zero or diverge to positive infinity. We contain the finite sample bias and RMSE in Table A.5 and observe these two features for the long- and short-run parameters by letting the sample size increase from 100 to 2000, corroborating the convergence rate in Theorem 1 (*i*).

Second, we compute the standard t-test using the OLS estimator to compare its finite sample null distribution with the standard mixed-normal distribution. Theorem 1 (*i*) implies that the null limit distribution of the t-test is the standard mixed-normal. Figure A.1 (a) shows the QQ-plots between the t-tests defined by the long- and short-run OLS estimators and the standard normal distribution. The QQ-plots are obtained by letting $T = 3000$ and repeating independent experiments 50000 times. They are distributed around the 45-degree line, and this corroborates the distributional property given in Theorem 1 (*i*).

DGP Condition under Theorem 1 (*ii*) We next verify Theorem 1 (*ii*) in the same way. We generate data according to the DGP condition in Section A.4.1 by letting $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 0, -3/2, 1)$ and $(\omega_*, \varrho_*) = (1/2, 0)$. This parameter condition obeys the condition in Theorem 1 (*ii*).

We verify Theorem 1 (*ii*) in the same manner to the earlier case. The only difference is in the convergence rate of the OLS estimator. Theorem 1 (*ii*) implies that the convergence rate of $\widehat{\theta}_T^+$ is T, but the other convergence rates are the same as before. Therefore, we modify the finite sample bias and RMSE of $\widehat{\theta}_T^+$ into the following:

$$\text{Bias} = \frac{1}{m}\sum_{j=1}^{m} T(\widehat{\theta}_{T,j}^+ - \theta_*^+) \quad \text{and} \quad \text{RMSE} = \sqrt{\frac{1}{m}\sum_{j=1}^{m} T^2(\widehat{\theta}_{T,j}^+ - \theta_*^+)^2},$$

while maintaining the previous formula for the others. We contain the finite sample biases and RMSEs in Table A.6 and observe that they converge to zero and a positive constant as the sample size increases. This corroborates that the convergence rate given in Theorem 1 (*ii*) is correct.

Second, we compare the finite sample null distribution of the standard t-test with the standard mixed-normal distribution. Figure A.1 (b) shows the QQ-plots between the t-tests and the standard normal distribution. The QQ-plots are distributed around the 45-degree line, corroborating the distributional property given in Theorem 1 (*ii*).

DGP Condition under Theorem 1 (*iii*) We next verify Theorem 1 (*iii*) in the same way. We generate data according to the DGP condition in Section A.4.1 by letting $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 3/2, 0, 1)$ and $(\omega_*, \varrho_*) = (1/2, 0)$. This parameter condition obeys the condition in Theorem 1 (*iii*).

We verify Theorem 1 (*iii*) in a manner similar to Theorem 1 (*i*). Theorem 1 (*iii*) implies that the convergence rate of $\widehat{\theta}_T^-$ is T, and the other convergence rates are $\sqrt{T}$. Therefore, we modify the finite sample bias and RMSE of $\widehat{\theta}_T^-$ into the following:

$$\text{Bias} = \frac{1}{m}\sum_{j=1}^{m} T(\widehat{\theta}_{T,j}^- - \theta_*^-) \quad \text{and} \quad \text{RMSE} = \sqrt{\frac{1}{m}\sum_{j=1}^{m} T^2(\widehat{\theta}_{T,j}^- - \theta_*^-)^2},$$

while maintaining the same formula for the others as in Theorem 1 (*i*). We contain the finite sample biases and RMSEs in Table A.7 and corroborate that the convergence rate given in Theorem 1 (*iii*) is correct.

Second, we draw the QQ-plots between the finite sample null distribution of the standard t-tests and the standard mixed-normal distribution. Figure A.1 (c) shows the QQ-plots and corroborates the distributional property given in Theorem 1 (*iii*).

DGP Condition under Theorem 1 (*iv*) We finally verify Theorem 1 (*iv*). We generate data according to the DGP condition in Section A.4.1 by letting $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 3/2, -3/2, 0)$ and $(\omega_*, \varrho_*) = (1/2, 0)$ to obey the condition in Theorem 1 (*iv*).

Given that the convergence rate of the OLS estimator is $\sqrt{T}$, we compute the finite sample bias and RMSE as for Theorem 1 (*i*) and contain them in Table A.8. We observe that they converge to zero and positive constants, respectively, as T grows. Further, the QQ-plots between the t-tests and the standard mixed-normal are distributed according to the 45-degree line. These two features corroborate Theorem 1 (*iv*).

A.5 Empirical Supplements

In this section, we provide additional empirical supplements.

A.5.1 Preliminary Empirical Analysis

This section provides two preliminary empirical analyses. First, Table A.9 provides the descriptive statistics of the variables examined in Sections 7.2 and A.5.2. The sample period is from 1947Q1 to 2007:Q4.

Second, Table A.10 provides the testing results using Phillips and Perron's (1988) unit-root test applied to the partial sum processes for Tables 4 and A.11. As we apply the unit-root testing by including both constant and trend, or including only constant, two testing results are provided for each variable. Except for r_t defined as the tax-to-nominal GDP ratio as given in Section A.5.2, the test results show that nonstationary data analysis has to be conducted for the other variables.

A.5.2 Tax Changes Measured by Tax-to-GDP Ratio

This section extends the work of Romer and Romer (2010) to investigate the long- and short-run relationships between fiscal shocks and real GDP. Rather than using the tax change logarithm τ_t, we use $\Delta r_t := (\Delta T_{1t} + \Delta T_{2t})/NY_t$, which represents the tax change-to-nominal GDP ratio, to measure the effect of fiscal policy and to specify the models corresponding to (20), (21), (22), and (23). As with Tables 4 and 5, we estimate the models using OLS and 2SNARDL. The estimation and inference results are presented in Tables A.11 and A.12, respectively. We summarize the results as follows:

(a) For the exogenous tax change, the coefficient of y_{t-1} in Table A.11 is statistically significant at the 25% level for the NARDL and ARDL models, by the t-test. The F-test does not reject the hypothesis of no cointegration. However, the coefficient of u_{t-1} in the t-test of Table A.12 is statistically significant at the 25% and 10% levels for the NARDL and ARDL models, respectively. We give more weight to the inference results in Table A.12 than to those in Table A.11, because the long-run parameters can be estimated super-consistently by 2SNARDL. In addition, neither $\mathscr{W}_T^{(1)}$ nor $W_T^{(1)}$ rejects the symmetry hypothesis in the long-run parameters, indicating that the ARDL model can estimate a cointegrating relationship between y_t and r_t more efficiently.

(b) The ARDL model estimation indicates that an increase in an exogenous tax shock measured by r_{t-1} reduces the long-run log real GDP by about 3%. This is close to the result of Romer and Romer (2010), estimating that GDP will increase by approximately 3% over three years following a tax cut of 1% of GDP.

(c) The Wald tests $\mathscr{W}_T^{(2)}$ and $W_T^{(2)}$ do not reject the hypothesis of symmetric short-run parameters. Furthermore, neither $\mathscr{W}_T^{(3)}$ nor $W_T^{(3)}$ rejects the hypothesis of long- and short-run symmetry. This confirms that ARDL is appropriate for the long- and short-run relationships between y_t and r_t.

(d) For endogenous and aggregate tax changes, there is negligible evidence of cointegration between real GDP and the tax change. None of the t- or F-tests in Table A.11 rejects the hypothesis of no cointegration. Furthermore, none of the coefficients of u_{t-1} in Table A.12 is statistically significant. Specifically, for the aggregate tax change, the unit-root test does not affirm that r_t is nonstationary; see Section A.5.1. Therefore, we conclude that the long- and short-run relationships can be properly estimated only by the exogenous tax change. $\square$

In summary, our empirical results obtained using the specification in Romer and Romer (2010) provide qualitatively the same results as those in Section 7. In particular, the long-run relationship between y_t and r_t captured by the cointegration coefficient is close to their estimate.

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Estimation	Parameter \ T	100	500	1000	1500	2000
OLS	θ_*^+	0.0690	0.0152	0.0071	0.0051	0.0039
	θ_*^-	-0.0687	-0.0152	-0.0071	-0.0051	-0.0039
	ρ_*	-0.0557	-0.0105	-0.0048	-0.0035	-0.0026
	φ_*	0.0128	0.0024	0.0010	0.0007	0.0006
	π_*^+	-0.0019	-0.0002	0.0000	-0.0001	-0.0002
	π_*^-	0.0035	0.0005	-0.0005	0.0002	0.0000
2SNARDL	θ_*^+	-0.0573	-0.0200	-0.0099	-0.0070	-0.0054
	θ_*^-	0.0575	0.0200	0.0098	0.0070	0.0054
	ρ_*	-0.0447	-0.0049	-0.0012	-0.0008	-0.0004
	φ_*	0.0162	-0.0002	-0.0009	-0.0007	-0.0006
	π_*^+	0.0144	0.0091	0.0053	0.0037	0.0028
	π_*^-	-0.0135	-0.0088	-0.0058	-0.0036	-0.0030

Table A.1: Finite Sample Biases of OLS and 2SNADL Estimators. This table shows the finite sample biases of the OLS and 2SNARDL estimators. The total number of repetitions is 50000. DGP: $\Delta y_t = \rho_* u_{t-1} + \varphi_* \Delta y_{t-1} + \pi_*^+ \Delta x_t^+ + \pi_*^- \Delta x_t^- + e_t$, $u_t = y_t - \beta_*^+ x_t^+ - \beta_*^- x_t^- - \zeta_*(t-1)$, $\Delta x_t = 1/2 + v_t$, and $(e_t, v_t)' \sim$ IID $N(\mathbf{0}_2, \mathbf{I}_2)$ with $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 3/2, -3/2, 0)$.

Estimation	Parameter \ T	100	500	1000	1500	2000
OLS	θ_*^+	0.1518	0.0533	0.0360	0.0290	0.0258
	θ_*^-	0.2059	0.0583	0.0376	0.0300	0.0265
	ρ_*	0.1020	0.0353	0.0239	0.0193	0.0172
	φ_*	0.0870	0.0364	0.0256	0.0208	0.0185
	π_*^+	0.1647	0.0685	0.0479	0.0390	0.0348
	π_*^-	0.3081	0.1240	0.0867	0.0706	0.0628
2SNARDL	θ_*^+	0.1506	0.0549	0.0367	0.0295	0.0261
	θ_*^-	0.2330	0.0617	0.0388	0.0308	0.0270
	ρ_*	0.0999	0.0349	0.0238	0.0193	0.0172
	φ_*	0.0892	0.0368	0.0259	0.0209	0.0186
	π_*^+	0.1670	0.0688	0.0480	0.0391	0.0348
	π_*^-	0.3122	0.1235	0.0865	0.0705	0.0628

Table A.2: Finite Sample Root Mean Square Errors of OLS and 2SNADL Estimators. This table shows the finite sample root mean square errors of the OLS and 2SNARDL estimators. The total number of repetitions is 50000. DGP: $\Delta y_t = \rho_* u_{t-1} + \varphi_* \Delta y_{t-1} + \pi_*^+ \Delta x_t^+ + \pi_*^- \Delta x_t^- + e_t$, $u_t = y_t - \beta_*^+ x_t^+ - \beta_*^- x_t^- - \zeta_*(t-1)$, $\Delta x_t = 1/2 + v_t$, and $(e_t, v_t)' \sim$ IID $N(\mathbf{0}_2, \mathbf{I}_2)$ with $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 3/2, -3/2, 0)$.

Estimation	Parameter \ T	100	500	1000	1500	2000
OLS	θ_*^+	0.0733	0.0147	0.0073	0.0050	0.0042
	θ_*^-	-0.0709	-0.0147	-0.0074	-0.0050	-0.0042
	ρ_*	-0.0565	-0.0101	-0.0049	-0.0034	-0.0028
	φ_*	0.0060	0.0004	0.0002	0.0002	0.0002
	π_*^+	-0.0207	-0.0041	-0.0017	-0.0014	-0.0014
	π_*^-	0.0041	0.0005	0.0001	0.0007	0.0006
2SNARDL	θ_*^+	-0.0626	-0.0162	-0.0081	-0.0056	-0.0046
	θ_*^-	0.0668	0.0202	0.0104	0.0071	0.0060
	ρ_*	-0.0536	-0.0071	-0.0029	-0.0019	-0.0015
	φ_*	0.0187	0.0002	-0.0003	-0.0003	-0.0002
	π_*^+	0.0060	0.0056	0.0037	0.0024	0.0018
	π_*^-	-0.0110	-0.0074	-0.0044	-0.0024	-0.0021

Table A.3: FINITE SAMPLE BIASES OF OLS AND 2SNADL ESTIMATORS. This table shows the finite sample biases of the OLS and 2SNARDL estimators. The total number of repetitions is 50000. DGP: $\Delta y_t = \rho_* u_{t-1} + \varphi_* \Delta y_{t-1} + \pi_*^+ \Delta x_t^+ + \pi_*^- \Delta x_t^- + e_t$, $u_t = y_t - \beta_*^+ x_t^+ - \beta_*^- x_t^- - \zeta_*(t-1)$, $\Delta x_t = 1/4 + 1/2\Delta x_{t-1} + v_t$, and $(e_t, v_t)' \sim$ IID $N(\mathbf{0}_2, \mathbf{I}_2)$ with $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 3/2, -3/2, 0)$.

Estimation	Parameter \ T	100	500	1000	1500	2000
OLS	θ_*^+	0.1446	0.0503	0.0343	0.0277	0.0251
	θ_*^-	0.1936	0.0537	0.0355	0.0283	0.0256
	ρ_*	0.1000	0.0337	0.0229	0.0185	0.0168
	φ_*	0.0830	0.0345	0.0244	0.0200	0.0182
	π_*^+	0.1750	0.0707	0.0494	0.0401	0.0367
	π_*^-	0.3281	0.1253	0.0873	0.0708	0.0645
2SNARDL	θ_*^+	0.1406	0.0507	0.0345	0.0278	0.0252
	θ_*^-	0.2190	0.0561	0.0364	0.0288	0.0260
	ρ_*	0.1002	0.0332	0.0227	0.0184	0.0167
	φ_*	0.0854	0.0347	0.0246	0.0201	0.0182
	π_*^+	0.1755	0.0705	0.0494	0.0401	0.0367
	π_*^-	0.3304	0.1248	0.0871	0.0707	0.0644

Table A.4: FINITE SAMPLE ROOT MEAN SQUARE ERRORS OF OLS AND 2SNADL ESTIMATORS. This table shows the finite sample root mean square errors of the OLS and 2SNARDL estimators. The total number of repetitions is 50000. DGP: $\Delta y_t = \rho_* u_{t-1} + \varphi_* \Delta y_{t-1} + \pi_*^+ \Delta x_t^+ + \pi_*^- \Delta x_t^- + e_t$, $u_t = y_t - \beta_*^+ x_t^+ - \beta_*^- x_t^- - \zeta_*(t-1)$, $\Delta x_t = 1/4 + 1/2\Delta x_{t-1} + v_t$, and $(e_t, v_t)' \sim$ IID $N(\mathbf{0}_2, \mathbf{I}_2)$ with $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 3/2, -3/2, 0)$.

Estimation	Parameter	Con. Rate \ T	100	500	1000	1500	2000
BIAS	θ_*^+	$\sqrt{T}$	0.6620	0.3218	0.2360	0.1927	0.1694
	θ_*^-	$\sqrt{T}$	-0.6644	-0.3208	-0.2358	-0.1925	-0.1717
	ρ_*	$\sqrt{T}$	-0.5290	-0.2218	-0.1608	-0.1305	-0.1133
	φ_*	$\sqrt{T}$	0.0998	0.0478	0.0371	0.0358	0.0284
	π_*^+	$\sqrt{T}$	-0.0025	-0.0159	-0.0125	0.0074	0.0045
	π_*^-	$\sqrt{T}$	-0.0101	-0.0040	0.0099	-0.0087	0.0082
RMSE	θ_*^+	$\sqrt{T}$	1.4871	1.1821	1.1389	1.1252	1.1195
	θ_*^-	$\sqrt{T}$	2.0415	1.2909	1.1944	1.1619	1.1475
	ρ_*	$\sqrt{T}$	0.9861	0.7842	0.7564	0.7480	0.7444
	φ_*	$\sqrt{T}$	0.8439	0.8086	0.8096	0.8051	0.8071
	π_*^+	$\sqrt{T}$	1.6542	1.5294	1.5141	1.5099	1.5103
	π_*^-	$\sqrt{T}$	3.0823	2.7777	2.7341	2.7314	2.7288

Table A.5: FINITE SAMPLE BIAS AND RMSE OF OLS ESTIMATOR. This table shows the finite sample bias and RMSE of the OLS estimator under the DGP condition of Theorem 1 (*i*). The total number of repetitions is 50000. DGP: $\Delta y_t = \rho_* u_{t-1} + \varphi_* \Delta y_{t-1} + \pi_*^+ \Delta x_t^+ + \pi_*^- \Delta x_t^- + e_t$, $u_t = y_t - \beta_*^+ x_t^+ - \beta_*^- x_t^- - \zeta_*(t-1)$, $\Delta x_t = 1/2 + v_t$, and $(e_t, v_t)' \sim$ IID $N(\mathbf{0}_2, \mathbf{I}_2)$ with $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 3/2, -3/2, 1)$.

Estimation	Parameter	Con. Rate \ T	100	500	1000	1500	2000
BIAS	θ_*^+	T	0.3992	0.1210	0.0944	0.0524	0.0279
	θ_*^-	$\sqrt{T}$	-0.5741	-0.3598	-0.2601	-0.2052	-0.1868
	ρ_*	$\sqrt{T}$	-0.4243	-0.2525	-0.1765	-0.1388	-0.1261
	φ_*	$\sqrt{T}$	0.1732	0.0997	0.0667	0.0579	0.0509
	π_*^+	$\sqrt{T}$	0.0050	-0.0005	0.0014	0.0004	-0.0119
	π_*^-	$\sqrt{T}$	0.0007	-0.0048	-0.0046	-0.0204	-0.0011
RMSE	θ_*^+	T	8.1992	7.7456	7.5341	7.4918	7.4571
	θ_*^-	$\sqrt{T}$	1.6527	1.3495	1.2543	1.2176	1.2066
	ρ_*	$\sqrt{T}$	0.9416	0.8319	0.8014	0.7881	0.7864
	φ_*	$\sqrt{T}$	0.8891	0.8642	0.8514	0.8458	0.8480
	π_*^+	$\sqrt{T}$	1.5746	1.5267	1.5126	1.5258	1.5086
	π_*^-	$\sqrt{T}$	2.8861	2.7729	2.7416	2.7369	2.7271

Table A.6: FINITE SAMPLE BIAS AND RMSE OF OLS ESTIMATOR. This table shows the finite sample bias and RMSE of the OLS estimator under the DGP condition of Theorem 1 (*ii*). The total number of repetitions is 50000. DGP: $\Delta y_t = \rho_* u_{t-1} + \varphi_* \Delta y_{t-1} + \pi_*^+ \Delta x_t^+ + \pi_*^- \Delta x_t^- + e_t$, $u_t = y_t - \beta_*^+ x_t^+ - \beta_*^- x_t^- - \zeta_*(t-1)$, $\Delta x_t = 1/2 + v_t$, and $(e_t, v_t)' \sim$ IID $N(\mathbf{0}_2, \mathbf{I}_2)$ with $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 0, -3/2, 1)$.

Estimation	Parameter	Con. Rate \ T	100	500	1000	1500	2000
BIAS	θ_*^+	$\sqrt{T}$	0.5713	0.2648	0.1867	0.1564	0.1339
	θ_*^-	T	-0.5093	-0.0685	-0.1303	-0.0330	-0.0829
	ρ_*	$\sqrt{T}$	-0.4594	-0.1827	-0.1256	-0.1052	-0.0904
	φ_*	$\sqrt{T}$	0.0750	0.0339	0.0188	0.0217	0.0176
	π_*^+	$\sqrt{T}$	-0.0229	-0.0083	-0.0085	-0.0036	-0.0005
	π_*^-	$\sqrt{T}$	0.0118	0.0079	-0.0168	0.0089	-0.0082
RMSE	θ_*^+	$\sqrt{T}$	1.4031	1.0743	1.0309	1.0214	1.0147
	θ_*^-	T	16.300	13.829	13.557	13.417	13.414
	ρ_*	$\sqrt{T}$	0.9015	0.7060	0.6809	0.6758	0.6731
	φ_*	$\sqrt{T}$	0.8270	0.7800	0.7828	0.7765	0.7783
	π_*^+	$\sqrt{T}$	1.6381	1.5289	1.5194	1.5082	1.5083
	π_*^-	$\sqrt{T}$	3.0672	2.7835	2.7405	2.7345	2.7232

Table A.7: Finite Sample Bias and RMSE of OLS Estimator. This table shows the finite sample bias and RMSE of the OLS estimator under the DGP condition of Theorem 1 (*iii*). The total number of repetitions is 50000. DGP: $\Delta y_t = \rho_* u_{t-1} + \varphi_* \Delta y_{t-1} + \pi_*^+ \Delta x_t^+ + \pi_*^- \Delta x_t^- + e_t$, $u_t = y_t - \beta_*^+ x_t^+ - \beta_*^- x_t^- - \zeta_*(t-1)$, $\Delta x_t = 1/2 + v_t$, and $(e_t, v_t)' \sim$ IID $N(\mathbf{0}_2, \mathbf{I}_2)$ with $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 3/2, 0, 1)$.

Estimation	Parameter	Con. Rate \ T	100	500	1000	1500	2000
BIAS	θ_*^+	$\sqrt{T}$	0.6768	0.3285	0.2547	0.1969	0.1702
	θ_*^-	$\sqrt{T}$	-0.6737	-0.3247	-0.2561	-0.1951	-0.1724
	ρ_*	$\sqrt{T}$	-0.5513	-0.2288	-0.1751	-0.1325	-0.1143
	φ_*	$\sqrt{T}$	0.1251	0.0525	0.0430	0.0280	0.0258
	π_*^+	$\sqrt{T}$	-0.0214	-0.0032	-0.0088	-0.0020	-0.0146
	π_*^-	$\sqrt{T}$	0.0024	0.0173	-0.0112	0.0022	0.0142
RMSE	θ_*^+	$\sqrt{T}$	1.5038	1.1912	1.1460	1.1318	1.1174
	θ_*^-	$\sqrt{T}$	2.0446	1.2982	1.2131	1.1678	1.1444
	ρ_*	$\sqrt{T}$	1.0148	0.7887	0.7605	0.7527	0.7437
	φ_*	$\sqrt{T}$	0.8654	0.8137	0.8136	0.8055	0.8091
	π_*^+	$\sqrt{T}$	1.6534	1.5324	1.5218	1.5174	1.5112
	π_*^-	$\sqrt{T}$	3.0824	2.7782	2.7496	2.7250	2.7190

Table A.8: Finite Sample Bias and RMSE of OLS Estimator. This table shows the finite sample bias and RMSE of the OLS estimator under the DGP condition of Theorem 1 (*iv*). The total number of repetitions is 50000. DGP: $\Delta y_t = \rho_* u_{t-1} + \varphi_* \Delta y_{t-1} + \pi_*^+ \Delta x_t^+ + \pi_*^- \Delta x_t^- + e_t$, $u_t = y_t - \beta_*^+ x_t^+ - \beta_*^- x_t^- - \zeta_*(t-1)$, $\Delta x_t = 1/2 + v_t$, and $(e_t, v_t)' \sim$ IID $N(\mathbf{0}_2, \mathbf{I}_2)$ with $(\rho_*, \pi_*^+, \pi_*^-, \beta_*^+, \beta_*^-, \zeta_*) = (-1/2, 1/2, -1/2, 3/2, -3/2, 0)$.

	Δy_t	Exo.			Endo.			Sum.		
		$\Delta\tau_{1t}$	$\Delta\tau_{2t}$	$\Delta\tau_t$	$\Delta\tau_{1t}$	$\Delta\tau_{2t}$	$\Delta\tau_t$	$\Delta\tau_{1t}$	$\Delta\tau_{2t}$	$\Delta\tau_t$
Mean	0.8256	0.4240	-0.4704	-0.0464	0.3773	-0.1841	0.1932	0.7564	-0.6228	0.1336
Median	0.7876	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Maximum	4.0198	6.4312	0.0000	6.4312	6.7617	0.0000	6.7617	6.7617	0.0000	6.7617
Minimum	-2.7525	0.0000	-7.0965	-7.0965	0.0000	-7.1122	-7.1122	0.0000	-7.1122	-7.1122
Std. Dev.	0.9780	1.3755	1.5073	2.1364	1.3317	1.0117	1.7136	1.7775	1.7316	2.6653
Skewness	-0.0501	3.0654	-3.0598	-0.2762	3.4094	-5.4276	0.4394	2.0432	-2.5281	-0.1450
Kurtosis	4.3614	10.8514	10.8759	6.3787	13.2115	31.1514	10.4245	5.4941	7.7131	4.1776
Obs.	243	243	243	243	243	243	243	243	243	243
	Δy_t	Exo. ratio			Endo. ratio			Sum. ratio		
		Δr_{1t}	Δr_{2t}	Δr_t	Δr_{1t}	Δr_{2t}	Δr_t	Δr_{1t}	Δr_{2t}	Δr_t
Mean	0.8256	0.0246	-0.0517	-0.0271	0.0471	-0.0260	0.0211	0.0679	-0.0738	-0.0059
Median	0.7876	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Maximum	4.0198	0.6977	0.0000	0.6977	1.9542	0.0000	1.9542	1.9542	0.0000	1.9542
Minimum	-2.7525	0.0000	-1.8706	-1.8706	0.0000	-2.8214	-2.8214	0.0000	-2.8214	-2.8214
Std. Dev.	0.9780	0.0963	0.2171	0.2428	0.2245	0.2028	0.3066	0.2351	0.2876	0.3847
Skewness	-0.0501	4.6231	-5.3926	-3.7013	6.0062	-11.5395	-1.0526	5.1164	-5.9045	-1.3783
Kurtosis	4.3614	25.4621	35.2975	25.0905	41.7414	152.4675	44.1557	33.0743	45.6366	21.6860
Obs.	243	243	243	243	243	243	243	243	243	243

Table A.9: DESCRIPTIVE STATISTICS. This table shows the descriptive statistics used in Sections 7.2 and A.5.2.

PP test	y_t	Exo.			Endo.			Sum.		
		τ_{1t}	τ_{2t}	τ_t	τ_{1t}	τ_{2t}	τ_t	τ_{1t}	τ_{2t}	τ_t
PP test w/o trend	-1.4767	0.7551	-0.7188	-1.6243	-2.4701	-0.3499	-2.2743	-1.3064	-0.4173	-2.0303
p-value	0.5439	0.9931	0.8387	0.4686	0.1241	0.9140	0.1812	0.6270	0.9028	0.2738
PP test w/ trend	-2.3488	-1.8278	-1.3653	-1.3483	-0.7094	-2.2995	-0.8156	-0.9285	-2.0326	-2.7414
p-value	0.4056	0.6883	0.8686	0.8732	0.9707	0.4322	0.9618	0.9500	0.5801	0.2210
PP test	y_t	Exo. ratio			Endo. ratio			Sum. ratio		
		r_{1t}	r_{2t}	r_t	r_{1t}	r_{2t}	r_t	r_{1t}	r_{2t}	r_t
PP test w/o trend	-1.4767	0.7288	-0.7859	-1.7004	-2.5409	-0.9379	-3.0710	-1.7735	-1.8303	-2.9448
p-value	0.5439	0.9926	0.8210	0.4299	0.1071	0.7749	0.0301	0.3932	0.3652	0.0418
PP test w/ trend	-2.3488	-1.8466	-2.3558	-2.2056	-1.5667	-2.0336	-2.6499	-1.3375	-2.3690	-3.4114
p-value	0.4056	0.6790	0.4019	0.4840	0.8033	0.5796	0.2587	0.8761	0.3949	0.0521

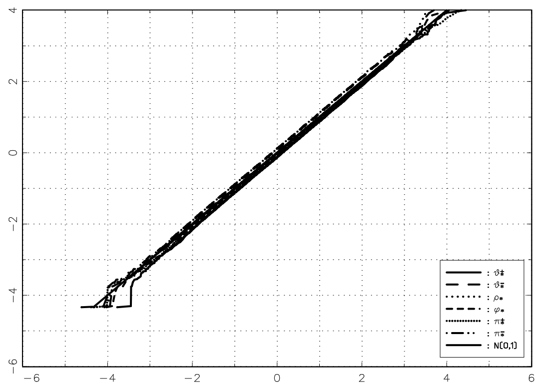
Table A.10: PHILLIPS AND PERRON'S (1988) UNIT-ROOT TESTS APPLIED TO THE QUARTERLY DATA IN ROMER AND ROMER (2010). Two Phillips and Perron's test statistics are computed using the data in Table A.9 by including the time trend and/or the constant. The lag lengths are selected by BIC.

NARDL Model				ARDL Model			
Variables \ Tax	Exo. ratio	Endo. ratio	Sum. ratio	Variables \ Tax	Exo. ratio	Endo. ratio	Sum. ratio
y_{t-1}	-0.0609†	-0.0633	-0.0460	y_{t-1}	-0.0552†	-0.0658	-0.0400
	(0.0184)	(0.0251)	(0.0177)		(0.0163)	(0.0247)	(0.0160)
r_{t-1}^{+}	-0.1992†	0.0850	0.0037	r_{t-1}	-0.1125†	0.1003	-0.0351
	(0.1349)	(0.0954)	(0.0789)		(0.0743)	(0.0925)	(0.0618)
r_{t-1}^{-}	-0.0970†	0.0401	-0.0566				
	(0.0801)	(0.1345)	(0.0645)				
Trend	0.0499***	0.0488**	0.0324**	Trend	0.0419***	0.0520***	0.0317***
	(0.0173)	(0.0200)	(0.0136)		(0.0127)	(0.0194)	(0.0135)
Constant	0.2683	0.3673†	0.3685*	Constant	0.4054**	0.4700***	0.5261***
	(0.2363)	(0.2380)	(0.2047)		(0.1787)	(0.1663)	(0.1543)
Δy_{t-1}	0.3095***	0.3144***	0.3049***	Δy_{t-1}	0.3042***	0.3115***	0.2921***
	(0.0639)	(0.0672)	(0.0657)		(0.0636)	(0.0660)	(0.0650)
Δy_{t-2}	0.1223*	0.1265*	0.1139*	Δy_{t-2}	0.1168*	0.1242*	0.1073*
	(0.0652)	(0.0660)	(0.0651)		(0.0644)	(0.0657)	(0.0648)
Δr_t^{+}	0.1638	0.2465	0.1778	Δr_t	0.1725	0.2553	0.1467
	(0.6215)	(0.2788)	(0.2621)		(0.6180)	(0.2776)	(0.2609)
Δr_t^{-}	-0.2706	-0.1477	-0.2350				
	(0.2734)	(0.3051)	(0.2087)				
AIC	2.6831	2.6925	2.6895	AIC	2.6739	2.6780	2.6814
BIC	2.8133	2.8226	2.8196	BIC	2.7751	2.7792	2.7826
t-test	-3.3087†	-2.5198	-2.5917	t-test	-3.3867†	-2.6602	-2.4960
F-test	3.9276	3.5933	3.4873	F-test	5.7417	5.2684	4.8210
$\mathscr{W}_T^{(1)}$					52.7574	2.1358	48.9740
					(0.1294)	(0.7290)	(0.2581)
$\mathscr{W}_T^{(2)}$					0.0722	0.8121	0.4782
					(0.8015)	(0.4021)	(0.4981)
$\mathscr{W}_T^{(3)}$					52.8296	2.9480	49.4522
					(0.1340)	(0.7595)	(0.2610)
$W_T^{(1)}$					0.5384	0.3344	0.8601
					(0.5697)	(0.6463)	(0.5035)
$W_T^{(2)}$					0.4079	0.9926	1.5118
					(0.5278)	(0.3177)	(0.2279)
$W_T^{(3)}$					1.0869	1.2200	2.1652
					(0.6618)	(0.6150)	(0.4681)

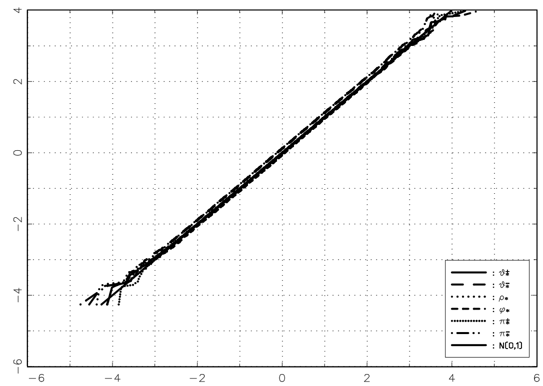
Table A.11: OLS Estimation of the NARDL and ARDL Models. This table presents the OLS estimation using quarterly data from Romer and Romer (2010). The left and right panels display estimated parameters for (23) and (21) using r_t instead of τ_t, respectively. Figures in parentheses indicate standard errors of the OLS estimates. At the bottom of the top panels, AIC, BIC, t-test, and Pesaran et al.'s (2001) F-test are reported. †, *, **, and *** indicate significance at 25%, 10%, 5%, and 1% levels, respectively. Wald tests in the last two bottom panels show the Wald tests in Section 5 and the standard Wald tests. Figures in parentheses below the Wald tests show p-values. They are obtained from 100000 bootstrap iterations.

	NARDL Model				ARDL Model			
	Variables \ Tax	Exo. ratio	Endo. ratio	Sum. ratio	Variables \ Tax	Exo. ratio	Endo. ratio	Sum. ratio
Long-Run	Constant	-2.5602	-2.5888**	-2.9531	Constant	-0.3468	-2.2311*	-1.0492
		(2.4603)	(1.1990)	(2.5202)		(2.3897)	(1.2927)	(2.8199)
	r^+_{t-1}	-6.7407***	4.2616***	4.5455***	r_{t-1}	-3.0752**	3.3960***	2.2608*
		(2.0794)	(0.5100)	(1.2686)		(1.2428)	(0.5463)	(1.2356)
	r^-_{t-1}	-2.8360*	5.1176***	2.7623**				
		(1.5211)	(0.9583)	(1.2373)				
	Trend	0.8364***	0.8391***	0.8435***	Trend	0.8242***	0.8370***	0.8295***
		(0.0166)	(0.0081)	(0.0170)		(0.0161)	(0.0087)	(0.0190)
Short-Run	u_{t-1}	-0.0609†	-0.0633	-0.0460	u_{t-1}	-0.0558*	-0.0627	-0.0378
		(0.0184)	(0.0251)	(0.0177)		(0.0162)	(0.0248)	(0.0160)
	Constant	0.6763***	0.6693***	0.6719***	Constant	0.5730***	0.6828***	0.6517***
		(0.1484)	(0.1510)	(0.1534)		(0.1470)	(0.1468)	(0.1464)
	Δy_{t-1}	0.3095***	0.3144***	0.3049***	Δy_{t-1}	0.3038***	0.3127***	0.3007***
		(0.0639)	(0.0672)	(0.0657)		(0.0635)	(0.0669)	(0.0656)
	Δy_{t-2}	0.1223*	0.1265*	0.1139*	Δy_{t-2}	0.1149*	0.1220*	0.1070†
		(0.0652)	(0.0660)	(0.0651)		(0.0644)	(0.0658)	(0.0648)
	Δr^+_t	0.1638	0.2465	0.1778	Δr_t	-0.2094	0.0801	-0.0737
		(0.6215)	(0.2788)	(0.2621)		(0.2435)	(0.2087)	(0.1596)
	Δr^-_t	-0.2706	-0.1477	-0.2350				
		(0.2734)	(0.3051)	(0.2087)				

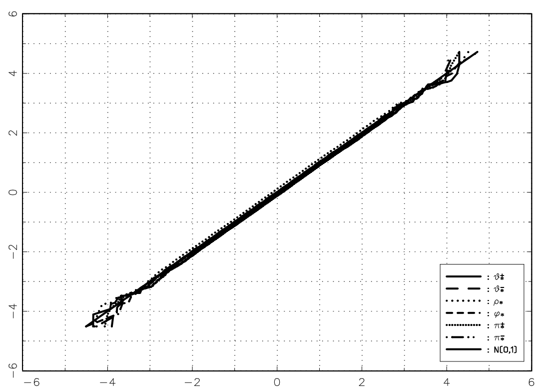
Table A.12: 2SNARDL ESTIMATION OF THE NARDL AND ARDL MODELS. This table presents the 2SNARDL estimation using the quarterly data from Romer and Romer (2010). The left and right panels display estimated parameters for (22) and (20), respectively. †, *, **, and *** imply that the tests are significant at 25%, 10%, 5%, and 1%, respectively.



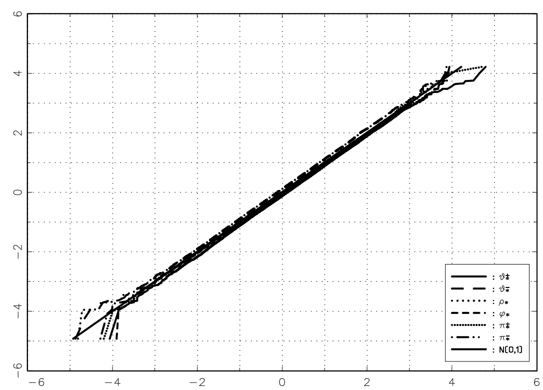
(a) DGP under Theorem 1 (*i*)



(b) DGP under Theorem 1 (*ii*)



(c) DGP under Theorem 1 (*iii*)



(d) DGP under Theorem 1 (*iv*)

Figure A.1: QQ-PLOTS OF THE t-TESTS UNDER THE NULL AND DGP CONDITIONS IN THEOREM 1. This figure shows the QQ-plots of the t-tests defined by $\widehat{\theta}_T^+$, $\widehat{\theta}_T^-$, $\widehat{\rho}_T$, $\widehat{\varphi}_T$, $\widehat{\pi}_T^+$, and $\widehat{\pi}_T^-$ under the null hypothesis and the DGP conditions in Theorem 1. To draw the QQ-plots, we let $T = 3000$ and computed the standard t-test using the OLS estimator. Total number of independent experiments is 50000.