Mathematical Proofs for "Testing for Neglected Nonlinearity Using Twofold Unidentified Models under the Null and Hexic Expansions"

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Abstract

We provide mathematical proofs for the results in "Testing for Neglected Nonlinearity Using Twofold Unidentified Models under the Null and Hexic Expansions" by Cho, Ishida, and White (2013).

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1 Introduction

This note provides mathematical proofs of the results stated Cho, Ishida, and White (2013). We avoid possible confusions by indicating the equation numbers in Cho, Ishida, and White (2013) using square brackets.

2 Appendix

Proof of Lemma 2: (i) We note that

$$(1) \quad \boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU} = \sum_{t=1}^{n}(\sum_{j=1}^{k} X_{t,j}d_j)^3 U_t - \sum_{t=1}^{n}(\sum_{j=1}^{k} X_{t,j}d_j)^3 \mathbf{Z}_t'(\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}_t')^{-1}\sum_{t=1}^{n}\mathbf{Z}_tU_t,$$

and that

$$\sum_{t=1}^{n}(\sum_{j=1}^{k} X_{t,j}d_j)^3 U_t = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{\ell=1}^{k} d_id_jd_\ell\left(\sum_{t=1}^{n} X_{t,i}X_{t,j}X_{t,\ell}U_t\right),$$

and

$$\sum_{t=1}^{n}(\sum_{j=1}^{k} X_{t,j}d_j)^3 \mathbf{Z}_t = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{\ell=1}^{k} d_id_jd_\ell\left(\sum_{t=1}^{n} X_{t,i}X_{t,j}X_{t,\ell}\mathbf{Z}_t\right).$$

We can also apply the CLT to $\sum_{t=1}^{n} X_{t,i}X_{t,j}X_{t,\ell}U_t$ and $\sum_{t=1}^{n}\mathbf{Z}_tU_t$ under Assumption 5, which imposes the finite moment condition on $|X_{t,j}|$ and $|U_t|$ to apply McLeish's (1974, Theorem 2.3) CLT on MDA. Therefore, we now have

$$\left\{\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\mathbf{Z}_tU_t, \frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_{t,i}X_{t,j}X_{t,\ell}U_t : i,j,\ell = 1,2,\ldots,k\right\} \Rightarrow \{\boldsymbol{\mathcal{Z}}, \mathcal{Z}_{i,j,\ell} : i,j,\ell = 1,2,\ldots,k\},$$

where $\boldsymbol{\mathcal{Z}}$ and $\mathcal{Z}_{i,j,\ell}$ are mean-zero normal random variables such that $E[\boldsymbol{\mathcal{Z}}\boldsymbol{\mathcal{Z}}'] = E[U_t^2\mathbf{Z}_t\mathbf{Z}_t']$, and for each $i,j,\ell,i',j',\ell' = 1,2,\ldots,k$, $E[\mathcal{Z}_{i,j,\ell}\boldsymbol{\mathcal{Z}}] = E[U_t^2X_{t,i}X_{t,j}X_{t,\ell}\mathbf{Z}_t]$, and $E[\mathcal{Z}_{i,j,\ell}\mathcal{Z}_{i',j',\ell'}] = E[U_t^2X_{t,i}X_{t,j}X_{t,\ell}X_{t,i'}X_{t,j'}X_{t,\ell'}]$. In particular, the weak limits are non-degenerate by Assumption 5(iii). In addition, we can apply the ergodic theorem to the other components. In other words,

$$(2) \quad \frac{1}{n}\sum_{t=1}^{n} X_{t,i}X_{t,j}X_{t,\ell}\mathbf{Z}_t \overset{\mathbb{P}}{\to} \boldsymbol{\omega}_{i,j,\ell} := E[X_{t,i}X_{t,j}X_{t,\ell}\mathbf{Z}_t] \quad \text{and} \quad \frac{1}{n}\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}_t' \overset{\mathbb{P}}{\to} E[\mathbf{Z}_t\mathbf{Z}_t']$$

under Assumptions 1 and 5. Given the DGP and the moment conditions, this result easily follows by the ergodic theorem. Thus, for each $\boldsymbol{d} \in \mathbb{S}^{k-1}$,

$$\frac{1}{\sqrt{n}}\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU} = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{\ell=1}^{k} d_jd_id_\ell\frac{1}{\sqrt{n}}\sum_{t=1}^{n} U_tV_{t,ji\ell}^{*} + o_{\mathbb{P}}(1) \Rightarrow \mathcal{G}_2(\boldsymbol{d}).$$

The weak limit is $O_{\mathbb{P}}(1)$ uniformly in $\boldsymbol{d}$ by the fact that for each $j = 1, 2, \ldots, k$, $|d_j| \leq 1$ and k is finite.

We next show that $\{n^{-1/2}\boldsymbol{\iota}'\mathbf{D}_3(\cdot)\mathbf{MU}\}$ is tight as given in Billingsley (1999) and van der Vaart and Wellner (1996). For this purpose, we show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{\|\boldsymbol{d}-\widetilde{\boldsymbol{d}}\|<\delta} \left| \frac{1}{\sqrt{n}}\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU} - \frac{1}{\sqrt{n}}\boldsymbol{\iota}'\mathbf{D}_3(\widetilde{\boldsymbol{d}})\mathbf{MU} \right| > \varepsilon \right) < \varepsilon.$$

Here, we note that

$$\left| \frac{1}{\sqrt{n}}\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU} - \frac{1}{\sqrt{n}}\boldsymbol{\iota}'\mathbf{D}_3(\widetilde{\boldsymbol{d}})\mathbf{MU} \right| \leq \left| \sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{\ell=1}^{k}(d_j d_i d_\ell - \widetilde{d}_j\widetilde{d}_i\widetilde{d}_\ell)W_{n,ji\ell} \right|$$
$$+ \left| \sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{\ell=1}^{k}(d_j d_i d_\ell - \widetilde{d}_j\widetilde{d}_i\widetilde{d}_\ell)\mathbf{W}_{n,ji\ell}\left(\frac{1}{n}\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}_t'\right)^{-1}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\mathbf{Z}_t Ut \right|,$$

where we let

$$W_{n,ji\ell} := \frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_{t,j}X_{t,i}X_{t,\ell}U_t \quad \text{and} \quad \mathbf{W}_{n,ji\ell} := \frac{1}{n}\sum_{t=1}^{n} X_{t,j}X_{t,i}X_{t,\ell}\mathbf{Z}_t,$$

for notational convenience. In addition, we note that

$$d_j d_i d_\ell - \widetilde{d}_j\widetilde{d}_i\widetilde{d}_\ell = d_j d_i(d_\ell - \widetilde{d}_\ell) + d_j(d_i - \widetilde{d}_i)\widetilde{d}_\ell + (d_j - \widetilde{d}_j)\widetilde{d}_i\widetilde{d}_\ell,$$

so that

$$\left| \sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{\ell=1}^{k}(d_j d_i d_\ell - \widetilde{d}_j\widetilde{d}_i\widetilde{d}_\ell)W_{n,ji\ell} \right| \leq \left| \sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{\ell=1}^{k} d_j d_i(d_\ell - \widetilde{d}_\ell)W_{n,ji\ell} \right|$$
$$+ \left| \sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{\ell=1}^{k} d_j(d_i - \widetilde{d}_i)\widetilde{d}_\ell W_{n,ji\ell} \right| + \left| \sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{\ell=1}^{k}(d_j - \widetilde{d}_j)\widetilde{d}_i\widetilde{d}_\ell W_{n,ji\ell} \right|.$$

Further,

$$\left| \sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{\ell=1}^{k} d_j d_i(d_\ell - \widetilde{d}_\ell)W_{n,ji\ell} \right| \leq \sum_{\ell=1}^{k}|d_\ell - \widetilde{d}_\ell| \sum_{j=1}^{k}\sum_{i=1}^{k}|W_{n,ji\ell}|,$$

$$\left| \sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{\ell=1}^{k} d_j(d_i - \widetilde{d}_i)\widetilde{d}_\ell W_{n,ji\ell} \right| \leq \sum_{i=1}^{k}|d_i - \widetilde{d}_i| \sum_{j=1}^{k}\sum_{\ell=1}^{k}|W_{n,ji\ell}|, \quad \text{and}$$

$$\left| \sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{\ell=1}^{k}(d_j - \widetilde{d}_j)\widetilde{d}_i\widetilde{d}_\ell W_{n,ji\ell} \right| \leq \sum_{j=1}^{k}|d_j - \widetilde{d}_j| \sum_{i=1}^{k}\sum_{\ell=1}^{k}|W_{n,ji\ell}|.$$

This implies that

$$\left|\sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{\ell=1}^{k}(d_jd_id_\ell - \widetilde{d}_j\widetilde{d}_i\widetilde{d}_\ell)W_{n,ji\ell}\right| \leq 3\sum_{\ell=1}^{k}|d_\ell - \widetilde{d}_\ell|\sum_{j=1}^{k}\sum_{i=1}^{k}|W_{n,ji\ell}|.$$

Similarly, it is not difficult to show that

$$\left|\sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{\ell=1}^{k}(d_jd_id_\ell - \widetilde{d}_j\widetilde{d}_i\widetilde{d}_\ell)\mathbf{W}_{n,ji\ell}'\left(\frac{1}{n}\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}_t'\right)^{-1}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\mathbf{Z}_tUt\right|$$
$$\leq 3\sum_{\ell=1}^{k}|d_\ell - \widetilde{d}_\ell|\sum_{j=1}^{k}\sum_{i=1}^{k}\left|\mathbf{W}_{n,ji\ell}'\left(\frac{1}{n}\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}_t'\right)^{-1}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\mathbf{Z}_tU_t\right|.$$

This now implies that

$$\left|\frac{1}{\sqrt{n}}\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU} - \frac{1}{\sqrt{n}}\boldsymbol{\iota}'\mathbf{D}_3(\widetilde{\boldsymbol{d}})\mathbf{MU}\right|$$
$$\leq 3\sum_{\ell=1}^{k}|d_\ell - \widetilde{d}_\ell|\sum_{j=1}^{k}\sum_{i=1}^{k}\left\{|W_{n,ji\ell}| + \left|\mathbf{W}_{n,ji\ell}'\left(\frac{1}{n}\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}_t'\right)^{-1}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\mathbf{Z}_tU_t\right|\right\}$$

and also that for any $\boldsymbol{d}$ and $\widetilde{\boldsymbol{d}}$,

$$\mathbb{P}\left(\left|\frac{1}{\sqrt{n}}\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU} - \frac{1}{\sqrt{n}}\boldsymbol{\iota}'\mathbf{D}_3(\widetilde{\boldsymbol{d}})\mathbf{MU}\right| > \varepsilon\right)$$
$$\leq \mathbb{P}\left(3\sum_{\ell=1}^{k}|d_\ell - \widetilde{d}_\ell|\sum_{j=1}^{k}\sum_{i=1}^{k}\left\{|W_{n,ji\ell}| + \left|\mathbf{W}_{n,ji\ell}'\left(\frac{1}{n}\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}_t'\right)^{-1}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\mathbf{Z}_tU_t\right|\right\} > \varepsilon\right).$$

For notational simplicity, we also let

$$S_{n,\ell} := \sum_{j=1}^{k}\sum_{i=1}^{k}\left\{|W_{n,ji\ell}| + \left|\mathbf{W}_{n,ji\ell}'\left(\frac{1}{n}\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}_t'\right)^{-1}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\mathbf{Z}_tU_t\right|\right\}.$$

Given this and the assumptions, applying the ergodic theorem yields that for each $j, i, \ell = 1, 2, \ldots, k$, $\mathbf{W}_{n,ji\ell} \overset{\mathbb{P}}{\to} \boldsymbol{\omega}_{ji\ell}$ and $n^{-1}\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}_t' \overset{\mathbb{P}}{\to} E[\mathbf{Z}_t\mathbf{Z}_t']$, and the CLT on MDA also yields that $\{W_{n,ji\ell}, n^{-1/2}\sum_{t=1}^{n}\mathbf{Z}_tU_t : j, i, \ell = 1, 2, \ldots, k\} \Rightarrow \{\mathcal{W}_{ji\ell}, \mathcal{Z} : j, i, \ell = 1, 2, \ldots, k\}$. Further, Assumption 5($iii$) implies that $E[\mathbf{Z}_t\mathbf{Z}_t']^{-1}$ is well defined. Therefore, for each $\ell = 1, 2, \ldots, k$, $S_{n,\ell} = O_{\mathbb{P}}(1)$, and if we let $\mathbf{S}_n := [S_{n,1}, S_{n,2}, \ldots, S_{n,k}]'$ and $|\boldsymbol{d} - \widetilde{\boldsymbol{d}}| := [|d_1 - \widetilde{d}_1|, \ldots, |d_k - \widetilde{d}_k|]'$,

$$\mathbb{P}\left(\frac{1}{\sqrt{n}}\left|\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU} - \boldsymbol{\iota}'\mathbf{D}_3(\widetilde{\boldsymbol{d}})\mathbf{MU}\right| > \varepsilon\right) \leq \mathbb{P}\left(3|\boldsymbol{d} - \widetilde{\boldsymbol{d}}|'\mathbf{S}_n > \varepsilon\right) \leq \mathbb{P}\left(3\|\boldsymbol{d} - \widetilde{\boldsymbol{d}}\|\|\mathbf{S}_n\| > \varepsilon\right),$$

where the last inequality holds by Cauchy-Schwarz's inequality. Given this, if $\delta \geq \|\boldsymbol{d} - \widetilde{\boldsymbol{d}}\|$,

$$\mathbb{P}\left(\sup_{\|\boldsymbol{d}-\widetilde{\boldsymbol{d}}\|<\delta} 3\|\boldsymbol{d} - \widetilde{\boldsymbol{d}}\|\|\mathbf{S}_n\| > \varepsilon\right) \leq \mathbb{P}\left(\|\mathbf{S}_n\| > \frac{\varepsilon}{3\delta}\right),$$

so that if we choose δ to be small enough, it is not difficult to show that $\limsup_{n\to\infty}\mathbb{P}(\|\mathbf{S}_n\| > \varepsilon/(3\delta)) < \varepsilon$. Thus, the tightness follows from this, and this implies that

$$(3)\quad n^{-1/2}\boldsymbol{\iota}'\mathbf{D}_3(\cdot)\mathbf{MU} \Rightarrow \mathcal{G}_2(\cdot).$$

Next, we note that

$$\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MD}_3(\boldsymbol{d})\boldsymbol{\iota} = \sum_{t=1}^{n}(\sum_{j=1}^{k}X_{t,j}d_j)^6 - \sum_{t=1}^{n}(\sum_{j=1}^{k}X_{t,j}d_j)^3\mathbf{Z}_t'(\sum_{t=1}^{n}\mathbf{Z}_t\mathbf{Z}_t')^{-1}\sum_{t=1}^{n}(\sum_{j=1}^{k}X_{t,j}d_j)^3\mathbf{Z}_t.$$

Here, we can apply the ergodic theorem easily. For this application, we note that

$$\sum_{t=1}^{n}(\sum_{j=1}^{k}X_{t,j}d_j)^6 = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{\ell=1}^{k}\sum_{i'=1}^{k}\sum_{j'=1}^{k}\sum_{\ell'=1}^{k}d_id_jd_\ell d_{i'}d_{j'}d_{\ell'}\left(\sum_{t=1}^{n}X_{t,i}X_{t,j}X_{t,\ell}X_{t,i'}X_{t,j'}X_{t,\ell'}\right),$$

so that when the finite moment condition in Assumption 5(i) holds,

$$\frac{1}{n}\sum_{t=1}^{n}X_{t,i}X_{t,j}X_{t,\ell}X_{t,i'}X_{t,j'}X_{t,\ell'} \overset{\mathbb{P}}{\to} E[X_{t,i}X_{t,j}X_{t,\ell}X_{t,i'}X_{t,j'}X_{t,\ell'}].$$

This fact and eq.(2) further imply that for each $\boldsymbol{d}\in\mathbb{S}^{k-1}$,

$$(4)\quad n^{-1}\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MD}_3(\boldsymbol{d})\boldsymbol{\iota} \overset{\mathbb{P}}{\to} \mathcal{J}_2(\boldsymbol{d},\boldsymbol{d}).$$

We here note that the ergodic theorem applies without associating it with its coefficient $d_id_jd_\ell d_{i'}d_{j'}d_{\ell'}$. Therefore, we can also claim the ULLN for the convergence in eq. (4), mainly because the space of $\boldsymbol{d}$ is a bounded unit circle, and k is finite. That is,

$$(5)\quad \sup_{\boldsymbol{d}\in\mathbb{S}^{k-1}}\left|\frac{1}{n}\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MD}_3(\boldsymbol{d})\boldsymbol{\iota} - \mathcal{J}_2(\boldsymbol{d},\boldsymbol{d})\right| \overset{\mathbb{P}}{\to} 0.$$

Finally, we can combine eqs. (3) and (5) by the converging-together-lemma.

$$(6)\quad \{n^{-1/2}\boldsymbol{\iota}'\mathbf{D}_3(\cdot)\mathbf{MU}, n^{-1}\boldsymbol{\iota}'\mathbf{D}_3(\cdot)\mathbf{MD}_3(\cdot)\boldsymbol{\iota}\} \Rightarrow \{\mathcal{G}_2(\cdot),\mathcal{J}_2(\cdot,\cdot)\}.$$

(ii) We can derive similar results for other partial derivatives. That is, the moment condition given by Assumption 5(i) is sufficient for McLeish's (1974) CLT on MDA, so that all of $n^{-1/2}\boldsymbol{\iota}'\mathbf{D}_5(\cdot)\mathbf{MU}$, $n^{-1/2}\boldsymbol{\iota}'\mathbf{D}_4(\cdot)\mathbf{MU}$, and $n^{-1/2}\boldsymbol{\iota}'\mathbf{D}_3(\cdot)\mathbf{MU}$ are $O_{\mathbb{P}}(1)$ as desired. ■

Proof of Theorem 2: We note that for each $\boldsymbol{d}\in\mathbb{S}^{k-1}$, if we let $\hat{h}_n(\boldsymbol{d})$ maximize the LHS of eq. [6], it asymptotically corresponds to maximizing the RHS of eq. [6] with respect to δ_n. We obtain the following result:

$$\frac{1}{\hat{\sigma}_{n,0}^2}\left\{L_n^{(2)}(\hat{h}_n(\boldsymbol{d})\boldsymbol{d};\lambda) - L_n^{(2)}(\mathbf{0};\lambda)\right\} = \frac{1}{\hat{\sigma}_{n,0}^2}\max\left[0, \frac{\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU}}{\sqrt{\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MD}_3(\boldsymbol{d})\boldsymbol{\iota}}}\right]^2 + o_{\mathbb{P}}(1)$$
$$\Rightarrow \max[0, \widetilde{\mathcal{G}}_0(\boldsymbol{d})]^2,$$

where for each $\boldsymbol{d} \in \mathbb{S}^{k-1}$, $\widetilde{\mathcal{G}}_0(\boldsymbol{d}) := \{\sigma_*^2\mathcal{J}_2(\boldsymbol{d},\boldsymbol{d})\}^{-1/2}\mathcal{G}_2(\boldsymbol{d})$, and the last weak convergence follows from the continuous mapping theorem. Thus, for each $\boldsymbol{d}$ and $\widetilde{\boldsymbol{d}}$, $E[\widetilde{\mathcal{G}}_0(\boldsymbol{d})] = 0$ and $E[\widetilde{\mathcal{G}}_0(\boldsymbol{d})\widetilde{\mathcal{G}}_0(\widetilde{\boldsymbol{d}})] = \widetilde{\rho}(\boldsymbol{d},\widetilde{\boldsymbol{d}})$. This is the desired result. ■

Proof of Lemma 3: (i) We focus on $\ell = 5$, as explained in the text. First, some tedious algebra shows that

$$(7)\quad \frac{\partial^5}{\partial h^5}N_n(h,\boldsymbol{d}) = 20\{\boldsymbol{\Psi}^{(3)}(h\boldsymbol{d})'\mathbf{MU}\}\{\boldsymbol{\Psi}^{(2)}(h\boldsymbol{d})'\mathbf{MU}\}$$
$$+ 10\{\boldsymbol{\Psi}^{(1)}(h\boldsymbol{d})'\mathbf{MU}\}\{\boldsymbol{\Psi}^{(4)}(h\boldsymbol{d})'\mathbf{MU}\} + 2\{\boldsymbol{\Psi}(h\boldsymbol{d})'\mathbf{MU}\}\{\boldsymbol{\Psi}^{(5)}(h\boldsymbol{d})'\mathbf{MU}\},$$

where for each m, $\boldsymbol{\Psi}^{(m)}(h\boldsymbol{d}) := (\partial^m/\partial h^m)\boldsymbol{\Psi}(h\boldsymbol{d})$. We also note that for each m, $\lim_{h\downarrow 0}\boldsymbol{\Psi}^{(m)}(h\boldsymbol{d})'$ $\mathbf{MU} = c_m\boldsymbol{\iota}'\mathbf{D}(\mathbf{0})\mathbf{MU}$ a.s.$-\mathbb{P}$, so that

$$\lim_{h\downarrow 0}\frac{\partial^5}{\partial h^5}N_n(h,\boldsymbol{d}) = 20c_2c_3\{\boldsymbol{\iota}'\mathbf{D}_2(\boldsymbol{d})\mathbf{MU}\}\{\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})'\mathbf{MU}\} + 10c_1c_4\{\boldsymbol{\iota}'\mathbf{D}_1(\boldsymbol{d})\mathbf{MU}\}\{\boldsymbol{\iota}'\mathbf{D}_4(\boldsymbol{d})\mathbf{MU}\}$$
$$+ 2c_0c_5\{\boldsymbol{\iota}'\mathbf{D}_0(\boldsymbol{d})\mathbf{MU}\}\{\boldsymbol{\iota}'\mathbf{D}_5(\boldsymbol{d})\mathbf{MU}\}$$

a.s.$-\mathbb{P}$. We further note that $c_2 = 0$, $\boldsymbol{\iota}'\mathbf{D}_1(\boldsymbol{d}) = \boldsymbol{d}'\mathbf{X}'$, and $\mathbf{D}_0(\boldsymbol{d}) = \mathbf{I}_n$, so that exploiting the fact that $\mathbf{M}$ is the idempotent matrix constructed by $\mathbf{Z} := [\boldsymbol{\iota}, \mathbf{X}]$ implies that $\lim_{h\downarrow 0}(\partial^5/\partial h^5)N_n(h,\boldsymbol{d})$ $= 0$ a.s. $-\mathbb{P}$. Second, we now examine the denominator. We note that some algebra yields that

$$(8)\quad \frac{\partial^5}{\partial h^5}D_n(h,\boldsymbol{d}) = 20\boldsymbol{\Psi}^{(3)}(h\boldsymbol{d})'\mathbf{M}\boldsymbol{\Psi}^{(2)}(h\boldsymbol{d}) + 10\boldsymbol{\Psi}^{(1)}(h\boldsymbol{d})'\mathbf{M}\boldsymbol{\Psi}^{(4)}(h\boldsymbol{d}) + 2\boldsymbol{\Psi}(h\boldsymbol{d})'\mathbf{M}\boldsymbol{\Psi}^{(5)}(h\boldsymbol{d}),$$

so that it now follows that

$$\lim_{h\downarrow 0}\frac{\partial^5}{\partial h^5}D_n(h,\boldsymbol{d}) = 20c_2c_3\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MD}_2(\boldsymbol{d})\boldsymbol{\iota} + 10c_1c_4\boldsymbol{\iota}'\mathbf{D}_1(\boldsymbol{d})\mathbf{MD}(\boldsymbol{d})\boldsymbol{\iota} + 2c_0c_5\boldsymbol{\iota}'\mathbf{D}_0(\boldsymbol{d})\mathbf{MD}_5(\boldsymbol{d})\boldsymbol{\iota}$$

a.s.$-\mathbb{P}$. From the fact that $c_2 = 0$ and $\mathbf{M}$ is the idempotent matrix constructed by $\mathbf{Z}$, it also trivially holds that $(\partial^5/\partial h^5)D_n(h,\boldsymbol{d}) = 0$ a.s. $-\mathbb{P}$.

(ii) We differentiate eq. (7) one more time with respect to h and obtain

$$(9)\quad \frac{\partial^6}{\partial h^6}N_n(h,\boldsymbol{d}) = 30\{\boldsymbol{\Psi}^{(4)}(h\boldsymbol{d})'\mathbf{MU}\}\{\boldsymbol{\Psi}^{(2)}(h\boldsymbol{d})'\mathbf{MU}\} + 20\{\boldsymbol{\Psi}^{(3)}(h\boldsymbol{d})'\mathbf{MU}\}\{\boldsymbol{\Psi}^{(3)}(h\boldsymbol{d})'\mathbf{MU}\}$$
$$+ 12\{\boldsymbol{\Psi}^{(1)}(h\boldsymbol{d})'\mathbf{MU}\}\{\boldsymbol{\Psi}^{(5)}(h\boldsymbol{d})'\mathbf{MU}\} + 2\{\boldsymbol{\Psi}(h\boldsymbol{d})'\mathbf{MU}\}\{\boldsymbol{\Psi}^{(6)}(h\boldsymbol{d})'\mathbf{MU}\}$$

and obtain

$$\lim_{h\downarrow 0}\frac{\partial^6}{\partial h^6}N_n(h,\boldsymbol{d}) = 30c_2c_4\{\boldsymbol{\iota}'\mathbf{D}_4(\boldsymbol{d})\mathbf{MU}\}\{\boldsymbol{\iota}'\mathbf{D}_2(\boldsymbol{d})\mathbf{MU}\} + 20c_3^2\{\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU}\}\{\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU}\}$$
$$+ 12c_1c_5\{\boldsymbol{\iota}'\mathbf{D}_1(\boldsymbol{d})\mathbf{MU}\}\{\boldsymbol{\iota}'\mathbf{D}_5(\boldsymbol{d})\mathbf{MU}\} + 2c_0c_6\{\boldsymbol{\iota}'\mathbf{D}_0(\boldsymbol{d})\mathbf{MU}\}\{\boldsymbol{\iota}'\mathbf{D}_6(\boldsymbol{d})\mathbf{MU}\}$$

a.s.$-\mathbb{P}$. We now note that $c_2 = 0$, $\boldsymbol{\iota}'\mathbf{D}_0(\boldsymbol{d})\mathbf{M} = \mathbf{0}$, and $\boldsymbol{\iota}'\mathbf{D}_1(\boldsymbol{d})\mathbf{M} = \mathbf{0}$, so that

$$\lim_{h\downarrow 0}\frac{\partial^6}{\partial h^6}N_n(h,\boldsymbol{d}) = 20c_3^2\{\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU}\}\{\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU}\} = 20c_3^2\{\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU}\}^2$$

a.s.$-\mathbb{P}$, as desired.

(*iii*) We again differentiate eq. (8) with respect to h and obtain

$$\frac{\partial^6}{\partial h^6}D_n(h,\boldsymbol{d}) = 30\{\boldsymbol{\Psi}^{(4)}(h\boldsymbol{d})'\mathbf{M}\boldsymbol{\Psi}^{(2)}(h\boldsymbol{d})\} + 20\{\boldsymbol{\Psi}^{(3)}(h\boldsymbol{d})'\mathbf{M}\boldsymbol{\Psi}^{(3)}(h\boldsymbol{d})\}$$
$$+ 12\{\boldsymbol{\Psi}^{(5)}(h\boldsymbol{d})'\mathbf{M}\boldsymbol{\Psi}^{(1)}(h\boldsymbol{d})\} + 2\{\boldsymbol{\Psi}^{(6)}(h\boldsymbol{d})'\mathbf{M}\boldsymbol{\Psi}(h\boldsymbol{d})\}$$

and from this,

$$\lim_{h\downarrow 0}\frac{\partial^6}{\partial h^6}D_n(h,\boldsymbol{d}) = 30c_2c_4\boldsymbol{\iota}'\mathbf{D}_4(\boldsymbol{d})\mathbf{M}\mathbf{D}_2(\boldsymbol{d})\boldsymbol{\iota} + 20c_3^2\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})'\mathbf{M}\mathbf{D}_3(\boldsymbol{d})\boldsymbol{\iota}$$
$$+ 12c_1c_5\boldsymbol{\iota}'\mathbf{D}_5(\boldsymbol{d})\mathbf{M}\mathbf{D}_1(\boldsymbol{d})\boldsymbol{\iota} + 2c_0c_6\boldsymbol{\iota}'\mathbf{D}_6(\boldsymbol{d})\mathbf{M}\mathbf{D}_0(\boldsymbol{d})\boldsymbol{\iota}.$$

We now note that $c_2 = 0$, $\boldsymbol{\iota}'\mathbf{D}_0(\boldsymbol{d})\mathbf{M} = \mathbf{0}$, and $\boldsymbol{\iota}'\mathbf{D}_1(\boldsymbol{d})\mathbf{M} = \mathbf{0}$, so that $\lim_{h\downarrow 0}(\partial^6/\partial h^6)D_n(h,\boldsymbol{d}) = 20c_3^2\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{M}\mathbf{D}_3(\boldsymbol{d})\boldsymbol{\iota}$ a.s.$-\mathbb{P}$. This is the desired result and completes the proof. ■

Proof of Lemma 4: We simply note that

$$\begin{aligned} QLR_n^{(1)} &= \sup_{\boldsymbol{d}\in\mathbb{S}^{k-1}}\sup_h \frac{\{\boldsymbol{\Psi}(h\boldsymbol{d})'\mathbf{MU}\}^2}{\hat{\sigma}_{n,0}^2\boldsymbol{\Psi}(h\boldsymbol{d})'\mathbf{M}\boldsymbol{\Psi}(h\boldsymbol{d})} \geq \sup_{\boldsymbol{d}\in\mathbb{S}^{k-1}}\lim_{h\downarrow 0}\frac{\{\boldsymbol{\Psi}(h\boldsymbol{d})'\mathbf{MU}\}^2}{\hat{\sigma}_{n,0}^2\boldsymbol{\Psi}(h\boldsymbol{d})'\mathbf{M}\boldsymbol{\Psi}(h\boldsymbol{d})} \\ &\overset{\text{a.s.}}{=} \sup_{\boldsymbol{d}\in\mathbb{S}^{k-1}}\frac{\{\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU}\}^2}{\hat{\sigma}_{n,0}^2\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{M}\mathbf{D}_3(\boldsymbol{d})\boldsymbol{\iota}} \geq \sup_{\boldsymbol{d}\in\mathbb{S}^{k-1}}\frac{1}{\hat{\sigma}_{n,0}^2}\max\left[0, \frac{\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{MU}}{\sqrt{\boldsymbol{\iota}'\mathbf{D}_3(\boldsymbol{d})\mathbf{M}\mathbf{D}_3(\boldsymbol{d})\boldsymbol{\iota}}}\right]^2 \\ &= QLR_n^{(2)} + o_{\mathbb{P}}(1). \end{aligned}$$

That is, $QLR_n^{(1)} \geq QLR_n^{(2)} + o_{\mathbb{P}}(1)$. This is the desired result. ■

Before proving Lemma 5, we first prove the following preliminary lemma, which is elementary by the given conditions. For notational simplicity, we let $\mathcal{T}_2^{(\ell,m)}(h\boldsymbol{d},\widetilde{h}\widetilde{\boldsymbol{d}}) := (\partial^{\ell+m}/\partial h^\ell\partial\widetilde{h}^m)$ $\mathcal{T}_2(h\boldsymbol{d},\widetilde{h}\widetilde{\boldsymbol{d}})$ and $\mathcal{J}_2^{(\ell)}(h\boldsymbol{d},h\boldsymbol{d}) := (\partial^\ell/\partial h^\ell)\mathcal{J}_2(h\boldsymbol{d},h\boldsymbol{d})$.

Lemma A1 *Given Assumptions 1, 2, 3, 6, 7, and $\mathcal{H}_0 : \lambda_* = 0$ or $\boldsymbol{\delta}_* = \mathbf{0}$, the following holds.*

(i) For $j = 0, 1, 2,$, $\lim_{h\downarrow 0} \mathcal{T}_1^{(j,0)}(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) = 0$*;*

(ii) $\lim_{h\downarrow 0} \mathcal{T}_1^{(3,0)}(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) = c_3 \mathcal{T}_3(\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}})$*;*

(iii) For $j = 0, 1, 2, 3, 4, 5$, $\lim_{h\downarrow 0} \mathcal{J}_1^{(j)}(h\boldsymbol{d}, h\boldsymbol{d}) = 0$*;*

(iv) $\lim_{h\downarrow 0} \mathcal{J}_1^{(6)}(h\boldsymbol{d}, h\boldsymbol{d}) = 20c_3^2 \mathcal{J}_2(\boldsymbol{d}, \boldsymbol{d})$*;*

(v) For $j = 0, 1, 2, 3, 4, 5, 6$, $\lim_{\widetilde{h}\downarrow 0} \lim_{h\downarrow 0} \mathcal{T}_1^{(j,0)}(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) = 0$*;*

(vi) For $j = 1, 2, 3, 4, 5$, $\lim_{\widetilde{h}\downarrow 0} \lim_{h\downarrow 0} \mathcal{T}_1^{(j,1)}(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) = 0$*;*

(vii) For $j = 2, 3, 4$, $\lim_{\widetilde{h}\downarrow 0} \lim_{h\downarrow 0} \mathcal{T}_1^{(j,2)}(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) = 0$*; and*

(viii) $\lim_{\widetilde{h}\downarrow 0} \lim_{h\downarrow 0} \mathcal{T}_1^{(3,3)}(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) = c_3^2 \mathcal{T}_2(\boldsymbol{d}, \widetilde{\boldsymbol{d}})$. □

Most parts follow by repeatedly using Lebesgue's dominated convergence theorem and the facts that $1 - E[\mathbf{Z}_t']E[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbf{Z}_t = 0$, $\mathbf{X}_t - E[\mathbf{X}_t\mathbf{Z}_t']E[\mathbf{Z}_t\mathbf{Z}_t']^{-1}\mathbf{Z}_t = \mathbf{0}$, and the condition that $c_2 = 0$. We omit the proofs for brevity.

Proof of Lemma 5: (i) For this proof, we apply Taylor's expansion with respect to h. In other words,

$$\mathcal{T}_1(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) = \lim_{h\downarrow 0} \sum_{j=0}^{3} \frac{1}{j!} \mathcal{T}_1^{(j)}(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) h^j + o(h^3) = \frac{c_3}{3!} \mathcal{T}_3(\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) h^3 + o(h^3)$$

by Lemma A1(i and ii) and

$$\mathcal{J}_1(h\boldsymbol{d}, h\boldsymbol{d}) = \lim_{h\downarrow 0} \sum_{j=0}^{6} \frac{1}{j!} \mathcal{J}_1^{(j)}(h\boldsymbol{d}, h\boldsymbol{d}) h^j + o(h^6) = \frac{20c_3^2}{6!} \mathcal{J}_2(\boldsymbol{d}, \boldsymbol{d}) h^6 + o(h^6) \tag{10}$$

by Lemma A1(v). This now implies that

$$\lim_{h\downarrow 0} \bar{\rho}(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) = \frac{c_3 \mathcal{T}_3(\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}})}{\{c_3^2 \sigma_*^2 \mathcal{J}_2(\boldsymbol{d}, \boldsymbol{d})\}^{1/2} \{\sigma_*^2 \mathcal{J}_1(\widetilde{\boldsymbol{d}}, \widetilde{\boldsymbol{d}})\}^{1/2}} = \text{sgn}[c_3] \widehat{\rho}(\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}),$$

as desired.

(ii) We again apply Taylor's expansion to $\mathcal{T}_1(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}})$ with respect to $(h, \widetilde{h})$. We note that

$$\begin{aligned} \mathcal{T}_1(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) &= \lim_{\widetilde{h}\downarrow 0} \lim_{h\downarrow 0} \sum_{i=0}^{6} \sum_{j=0}^{i} \frac{1}{i!} \binom{i}{j} \mathcal{T}_1^{(i-j,j)}(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) h^{i-1} \widetilde{h}^j + o((h^3 + \widetilde{h}^3)^2) \\ &= \frac{c_3^2}{6!} \binom{6}{3} \mathcal{T}_2(\boldsymbol{d}, \widetilde{\boldsymbol{d}}) h^3 \widetilde{h}^3 + o((h^3 + \widetilde{h}^3)^2) \end{aligned}$$

by Lemma A1(v, vi, vii, and $viii$). Using this lemma and eq. (10) yields

$$\lim_{\widetilde{h}\downarrow 0} \lim_{h\downarrow 0} \bar{\rho}(h\boldsymbol{d}, \widetilde{h}\widetilde{\boldsymbol{d}}) = \frac{c_3^2 \mathcal{T}_2(\boldsymbol{d}, \widetilde{\boldsymbol{d}})}{\{c_3^2 \sigma_*^2 \mathcal{J}_2(\boldsymbol{d}, \boldsymbol{d})\}^{1/2} \{c_3^2 \sigma_*^2 \mathcal{J}_2 \widetilde{\boldsymbol{d}}, \widetilde{\boldsymbol{d}})\}^{1/2}} = \widetilde{\rho}(\boldsymbol{d}, \widetilde{\boldsymbol{d}}).$$

This is the desired result and completes the proof. ■

References

BILLINGSLEY, P. (1999): *Convergence of Probability Measures*. New York: Wiley.

CHO, J., ISHIDA, I., AND WHITE, H. (2013): "Testing for Neglected Nonlinearity Using Twofold Unidentified Models under the Null and Hexic Expansions," Discussion Paper, School of Economics, Yonsei University.

MCLEISH, D. (1974): "Dependent Central Limit Theorem and Invariance Principles," *The Annals of Probability*, 2, 620–628.

VAN DER VAART, A. AND WELLER, J. (1996): *Weak Convergence and Empirical Processes with Applications to Statistics*. New York: Springer-Verlag.