Online Supplement for 'Functional Data Inference in a Parametric Quantile Model applied to Lifetime Income Curves'*

by

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This Online Supplement is an Appendix that provides proofs of all the results in the paper, including the lemmas, as well as some additional empirical findings. Proofs are given in Section A.1 and the supplementary empirical application is in Section A.2

A Appendix

A.1 Proofs

Proof of Lemma 1: Note that $d_\tau(\gamma, u) := \mathbb{E}[\xi_\tau(G(\gamma) - u)] - \mathbb{E}[\xi_\tau(G(\gamma) - x_\tau(\gamma))] = uF_\gamma(u) - uF_\gamma(x_\tau(\gamma)) + \int_{-\infty}^{x_\tau(\gamma)} gdF_\gamma(g) - \int_{-\infty}^{u} gdF_\gamma(g)$. Applying integration by parts, $x_\tau(\gamma)F_\gamma(x_\tau(\gamma)) = \int_{-\infty}^{x_\tau(\gamma)} F_\gamma(g)dg + \int_{-\infty}^{x_\tau(\gamma)} gdF_\gamma(g)$, and $uF_\gamma(u) = \int_{-\infty}^{u} F_\gamma(g)dg + \int_{-\infty}^{u} gdF_\gamma(g)$, giving $\int_{-\infty}^{x_\tau(\gamma)} g\ dF_\gamma(g) - \int_{-\infty}^{u} gdF_\gamma(g) = x_\tau(\gamma)F_\gamma(x_\tau(\gamma)) - uF_\gamma(u) + \int_{-\infty}^{x_\tau(\gamma)} F_\gamma(g)dg - \int_{-\infty}^{u} F_\gamma(g)dg$. Hence, $d_\tau(\gamma, u) = (x_\tau(\gamma) - u)F_\gamma(x_\tau(\gamma)) + \int_{-\infty}^{x_\tau(\gamma)} F_\gamma(g)dg - \int_{-\infty}^{u} F_\gamma(g)dg$. Further, if $x_\tau(\gamma) > u$, then $d_\tau(\gamma, u) = \int_{u}^{x_\tau(\gamma)}\{F_\gamma(x_\tau(\gamma)) - F_\gamma(g)\}dg$; and if $x_\tau(\gamma) < u$, then $d_\tau(\gamma, u) = \int_{x_\tau(\gamma)}^{u}\{F_\gamma(g) - F_\gamma(x_\tau(\gamma))\}dg$, so that $d_\tau(\gamma, u) := \mathbb{E}[\xi_\tau(G(\gamma) - u)] - \mathbb{E}[\xi_\tau(G(\gamma) - x_\tau(\gamma))] = \int_{\min[u, x_\tau(\gamma)]}^{\max[u, x_\tau(\gamma)]} |F_\gamma(g) - F_\gamma(x_\tau(\gamma))|dg$. This completes the proof. ■

Proof of Theorem 1: The proof follows reasoning similar to that of Oberhofer and Haupt (2016). Applying Lemma 2N of Oberhofer and Haupt (2016), we first obtain that for w such that $\|w\| = 1$,

$$\widehat{R}_{ln}(w) \le \frac{1}{\sqrt{n}} \sum_{i=1}^{n} w' \int_\gamma \nabla_{\theta_\tau} \rho_\tau(\gamma, \widehat{\theta}_{\tau n})(\mathbb{1}\{G_i(\gamma) \le \rho_\tau(\gamma, \widehat{\theta}_{\tau n})\} - \tau) d\mathbb{Q}(\gamma) \le \widehat{R}_{un}(w),$$

where

$$\widehat{R}_{ln}(w) := -\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_\gamma \mathbb{1}\{G_i(\gamma) = \rho_\tau(\gamma, \widehat{\theta}_{\tau n})\} |w' \nabla_{\theta_\tau} \rho_\tau(\gamma, \widehat{\theta}_{\tau n})| \mathbb{1}\{w' \nabla_{\theta_\tau} \rho_\tau(\gamma, \widehat{\theta}_{\tau n}) < 0\} d\mathbb{Q}(\gamma),$$

$$\widehat{R}_{un}(w) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_\gamma \mathbb{1}\{G_i(\gamma) = \rho_\tau(\gamma, \widehat{\theta}_{\tau n})\} |w' \nabla_{\theta_\tau} \rho_\tau(\gamma, \widehat{\theta}_{\tau n})| \mathbb{1}\{w' \nabla_{\theta_\tau} \rho_\tau(\gamma, \widehat{\theta}_{\tau n}) \ge 0\} d\mathbb{Q}(\gamma).$$

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Furthermore, applying Lemma 9N of Oberhofer and Haupt (2016) demonstrates that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w' \int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma,\widehat{\theta}_{\tau n})(\mathbb{1}\{G_i(\gamma)\leq \rho_\tau(\gamma,\widehat{\theta}_{\tau n})\}-\tau)d\mathbb{Q}(\gamma)$$
$$= w' A_\tau^* \sqrt{n}(\widehat{\theta}_{\tau n}-\theta_\tau^*) + w'\frac{1}{\sqrt{n}}\sum_{i=1}^{n} J_{\tau i} + o_{\mathbb{P}}(1),$$

so that if we show that $\widehat{R}_{ln}(w)=o_{\mathbb{P}}(1)$ and $\widehat{R}_{un}(w)=o_{\mathbb{P}}(1)$, then (3) follows. For this derivation, we let

$$\widehat{R}_n(w) := \widehat{R}_{ln}(w) - \widehat{R}_{un}(w) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_\gamma \mathbb{1}\{G_i(\gamma)=\rho_\tau(\gamma,\widehat{\theta}_{\tau n})\}|w'\nabla_{\theta_\tau}\rho_\tau(\gamma,\widehat{\theta}_{\tau n})|d\mathbb{Q}(\gamma)$$

and show that $\widehat{R}_n(w)=o_{\mathbb{P}}(1)$ by noting that $\widehat{R}_n(w)=o_{\mathbb{P}}(1)$ if and only if $\widehat{R}_{ln}(w)=o_{\mathbb{P}}(1)$ and $\widehat{R}_{un}(w)=o_{\mathbb{P}}(1)$. If we let $B(\theta_0,d) := \{\theta : \|\theta-\theta_0\|\leq d\}$, then for a sufficiently large $m<\infty$, there are finite numbers of open balls covering $\Theta(n,m) := \{\theta : \sqrt{n}\|\theta-\theta_\tau^*\|\leq m\}$, viz., $\Theta(n,m)\subset \cup_{j=1}^{n(d)} B(\theta(j,d),d)$ such that for any $d>0$, $n(d)<\infty$, where $\theta(j,d)$ is the center of the j-th open ball. Given this, Assumption 2 implies that for any $\theta_\tau\in B(\theta(j,d),d)$, there exists $\bar{\theta}_\tau$, θ_τ^a, and $\theta_\tau^b \in B(\theta(j,d),d)$ such that

$$\int_\gamma |w'\nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau)|^2 d\mathbb{Q}(\gamma) \leq \int_\gamma |w'\nabla_{\theta_\tau}\rho_\tau(\gamma,\bar{\theta}_\tau)|^2 d\mathbb{Q}(\gamma)$$

and

$$\int_\gamma \mathbb{1}\{G_i(\gamma)=\rho_\tau(\gamma,\theta_\tau)\}d\mathbb{Q}(\gamma)\leq \int_\gamma \mathbb{1}\{\rho_\tau(\gamma,\theta_\tau^a)\leq G_i(\gamma)\leq \rho_\tau(\gamma,\theta_\tau^b)\}d\mathbb{Q}(\gamma).$$

If we further let

$$\bar{R}_n(j,r) := \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_\gamma \mathbb{1}\{\rho_\tau(\gamma,\theta_\tau^a)\leq G_i(\gamma)\leq \rho_\tau(\gamma,\theta_\tau^b)\}d\mathbb{Q}(\gamma)\int_\gamma |w'\nabla_{\theta_\tau}\rho_\tau(\gamma,\bar{\theta}_\tau)|^2 d\mathbb{Q}(\gamma),$$

then it follows that $0\leq \widehat{R}_n(w)\leq \bar{R}_n(j,r)$ by noting that for any $\theta\in B(\theta(j,d),d)$,

$$\int_\gamma \mathbb{1}\{G_i(\gamma)=\rho_\tau(\gamma,\theta_\tau)\}|w'\nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau)|d\mathbb{Q}(\gamma)$$
$$\leq \int_\gamma \mathbb{1}\{\rho_\tau(\gamma,\theta_\tau^a)\leq G_i(\gamma)\leq \rho_\tau(\gamma,\theta_\tau^b)\}d\mathbb{Q}(\gamma)\int_\gamma |w'\nabla_{\theta_\tau}\rho_\tau(\gamma,\bar{\theta}_\tau)|^2 d\mathbb{Q}(\gamma).$$

We here note that for each $j=1,2,\ldots,n(d)$,

$$\mathbb{E}[\int_\gamma \mathbb{1}\{\rho_\tau(\gamma,\theta_\tau^a)\leq G_i(\gamma)\leq \rho_\tau(\gamma,\theta_\tau^b)\}d\mathbb{Q}(\gamma)\int_\gamma |w'\nabla_{\theta_\tau}\rho_\tau(\gamma,\bar{\theta}_\tau)|^2 d\mathbb{Q}(\gamma)]$$
$$= \int_\gamma \{F_\gamma(\rho_\tau(\gamma,\theta_\tau^b)) - F_\gamma(\rho_\tau(\gamma,\theta_\tau^a))\}d\mathbb{Q}(\gamma)\int_\gamma |w'\nabla_{\theta_\tau}\rho_\tau(\gamma,\bar{\theta}_\tau)|^2 d\mathbb{Q}(\gamma),$$

and

$$\text{var}[\bar{R}_n(j,r)] \leq \int_\gamma \{F_\gamma(\rho_\tau(\gamma,\theta_\tau^b)) - F_\gamma(\rho_\tau(\gamma,\theta_\tau^a))\}d\mathbb{Q}(\gamma)\left(\int_\gamma |w'\nabla_{\theta_\tau}\rho_\tau(\gamma,\bar{\theta}_\tau)|^2 d\mathbb{Q}(\gamma)\right)^2.$$

Further, for some $\widetilde{\theta}_\tau$ between θ_τ^a and θ_τ^b, we have

$$F_\gamma(\rho_\tau(\gamma,\theta_\tau^b)) - F_\gamma(\rho_\tau(\gamma,\theta_\tau^a)) = f_\gamma(\rho_\tau(\gamma,\widetilde{\theta}_\tau))\sqrt{n}(\theta_\tau^b - \theta_\tau^a)\frac{1}{\sqrt{n}}\nabla_{\theta_\tau}\rho_\tau(\gamma,\widetilde{\theta}_\tau) = o(1)$$

because $f_{(\cdot)}(\cdot)$ and $\nabla_{\theta_\tau}\rho_\tau(\cdot,\cdot)$ are uniformly bounded on $\Gamma \times \Theta_\tau$ by Assumption 2 and $\sqrt{n}(\theta_\tau^b - \theta_\tau^a) \leq 2d$. It therefore follows that $\bar{R}_n(j,r) = o_{\mathbb{P}}(1)$, so that

$$\mathbb{E}[\int_\gamma \mathbb{1}\{\rho_\tau(\gamma,\theta_\tau^a) \leq G_i(\gamma) \leq \rho_\tau(\gamma,\theta_\tau^b)\}d\mathbb{Q}(\gamma)\int_\gamma |w'\nabla_{\theta_\tau}\rho_\tau(\gamma,\bar{\theta}_\tau)|^2 d\mathbb{Q}(\gamma)] = o(1),$$

and $\text{var}[\bar{R}_n(j,r)] = o(1)$, leading to $\widehat{R}_n(w) = o_{\mathbb{P}}(1)$, and this again leads to (3).

Second, note that (3) implies that $\sqrt{n}(\widehat{\theta}_{\tau n} - \theta_\tau^*) = -A_\tau^{*-1}n^{-1/2}\sum_{i=1}^n J_{\tau i} + o_{\mathbb{P}}(1)$. Given that θ_τ^* is identified as given in Assumption 2, the first-order condition holds, so that $\mathbb{E}[J_{\tau i}] = 0$. Assumption 4 also implies that $B_\tau^* := \mathbb{E}[J_{\tau i}J_{\tau i}']$ is positive definite. Furthermore, Assumption 3 implies that for each $j = 1, 2, \ldots, c_\tau$, $\mathbb{E}[J_{\tau ij}^2] < \infty$, where $J_{\tau ij}$ is the j-th row element of $J_{\tau i}$. Therefore, $n^{-1/2}\sum_{i=1}^n J_{\tau i} \overset{\text{A}}{\sim} \mathcal{N}(0, B_\tau^*)$ by the multivariate CLT, so that $\sqrt{n}(\widehat{\theta}_{\tau n} - \theta_\tau^*) \overset{\text{A}}{\sim} \mathcal{N}(0, A_\tau^{*-1}B_\tau^*A_\tau^{*-1})$. This completes the proof. ■

Proof of Lemma 2: We show stochastic equicontinuity of $n^{-1/2}\sum_{i=1}^n(\mathbb{1}\{G_i(\cdot) \leq \rho_\tau(\cdot,\theta_\tau^0)\} - \tau)$ using Ossiander's L^2 entropy condition: for some $\nu > 0$ and $C > 0$,

$$\mathbb{E}\left(\sup_{\|\gamma-\gamma'\|<\delta}|\mathbb{1}\{G_i(\gamma) \leq \rho_\tau(\gamma,\theta_\tau^0)\} - \tau - (\mathbb{1}\{G_i(\gamma' \leq \rho_\tau(\gamma',\theta_\tau^0))\} - \tau)|^2\right)^{1/2} \leq C\delta^\nu.$$

To verify this, first note that if we let $U_i(\gamma) := F_\gamma(G_i(\gamma))$, where $F_\gamma(\cdot)$ is the CDF of $G_i(\gamma)$, the left side is identical to

$$\mathbb{E}\left(\sup_{\|\gamma-\gamma'\|<\delta}|\mathbb{1}\{U_i(\gamma) - \tau \leq 0\} - \mathbb{1}\{U_i(\gamma') - \tau \leq 0\})|^2\right)^{1/2}$$

by noting that $F_\gamma(\rho_\tau(\gamma,\theta_\tau^0)) = \tau$ and $F_{\gamma'}(\rho_\tau(\gamma',\theta_\tau^0)) = \tau$. Next, apply the proof in Andrews (1994, p. 2779), letting his U_t and $h^*(Z_t,\cdot)$ be 1 and $U_i(\cdot) - \tau$, respectively and note that Assumptions 1 and 2 imply that $U_i(\cdot)$ is Lipschitz continuous almost surely: for some $C > 0$, $|U_i(\gamma) - U_i(\gamma')| \leq C\|\gamma - \gamma'\|$. Here, we further note that $U_i(\gamma)$ is uniformly distributed over $[0,1]$, so that its density function is bounded above uniformly on Γ. Therefore, example 3 in Andrews (1994, p. 2779) proves equicontinuity by Ossiander's L^2 entropy condition.

Next derive the covariance structure of the Gaussian stochastic process $\mathcal{G}_\tau(\cdot)$, noting that for each γ and

γ',

$$\begin{aligned}
&\mathbb{E}[(\mathbb{1}\{G_i(\gamma) \le \rho_\tau(\gamma, \theta_\tau^0)\} - \tau)(\mathbb{1}\{G_i(\gamma') \le \rho_\tau(\gamma', \theta_\tau^0)\} - \tau)] \\
&\quad = \mathbb{E}[(\mathbb{1}\{U_i(\gamma) \le \tau\} - \tau)(\mathbb{1}\{U_i(\gamma') \le \tau\} - \tau)] \\
&\quad\quad = \mathbb{E}[\mathbb{1}\{U_i(\gamma) \le \tau\}\mathbb{1}\{U_i(\gamma') \le \tau\}] - \tau\mathbb{E}[\mathbb{1}\{U_i(\gamma) \le \tau\}] - \tau\mathbb{E}[\mathbb{1}\{U_i(\gamma') \le \tau\}] + \tau^2 \\
&\quad\quad\quad = \mathbb{E}[\mathbb{1}\{U_i(\gamma) \le \tau\}\mathbb{1}\{U_i(\gamma') \le \tau\}] - \tau^2 = \kappa(\gamma, \gamma'),
\end{aligned}$$

where the final equality follows from the fact that $\mathbb{E}[\mathbb{1}\{U_i(\gamma) \le \tau\}] = \tau$ uniformly on γ. This completes the proof. ■

Proof of Theorem 2: Given Lemma 2, we note by continuous mapping that

$$\int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma, \theta_\tau^0)\frac{1}{\sqrt{n}}\sum_{i=1}^n \left(\mathbb{1}\{G_i(\gamma) \le \rho_\tau(\gamma, \theta_\tau^0)\} - \tau\right) d\mathbb{Q}(\gamma) \Rightarrow \int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma, \theta_\tau^0)\mathcal{G}_\tau(\gamma)d\mathbb{Q}(\gamma)$$

which follows a normal distribution since $\mathcal{G}_\tau(\cdot)$ is a Gaussian stochastic process. Further note that applying dominated convergence using Assumption 3,

$$\mathbb{E}\left[\int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma, \theta_\tau^0)\mathcal{G}_\tau(\gamma)d\mathbb{Q}(\gamma)\right] = \int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma, \theta_\tau^0)\mathbb{E}[\mathcal{G}_\tau(\gamma)]d\mathbb{Q}(\gamma) = 0 \quad \text{and}$$

$$\begin{aligned}
&\mathbb{E}\left[\int_\gamma\int_{\gamma'} \nabla_{\theta_\tau}\rho_\tau(\gamma, \theta_\tau^0)\mathcal{G}_\tau(\gamma)\mathcal{G}_\tau(\gamma')\nabla_{\theta_\tau}\rho_\tau(\gamma', \theta_\tau^0)d\mathbb{Q}(\gamma)d\mathbb{Q}(\gamma')\right] \\
&\quad = \int_\gamma\int_{\gamma'} \nabla_{\theta_\tau}\rho_\tau(\gamma, \theta_\tau^0)\mathbb{E}[\mathcal{G}_\tau(\gamma)\mathcal{G}_\tau(\gamma')]\nabla'_{\theta_\tau}\rho_\tau(\gamma, \theta_\tau^0)d\mathbb{Q}(\gamma)d\mathbb{Q}(\gamma') \\
&\quad\quad = \int_\gamma\int_{\gamma'} \nabla_{\theta_\tau}\rho_\tau(\gamma, \theta_\tau^0)\kappa_\tau(\gamma, \gamma')\nabla'_{\theta_\tau}\rho_\tau(\gamma, \theta_\tau^0)d\mathbb{Q}(\gamma)d\mathbb{Q}(\gamma') =: B_\tau^0
\end{aligned}$$

by the definition of $\kappa_\tau(\cdot,\cdot)$. Therefore, $\int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma, \theta_\tau^0)\mathcal{G}_\tau(\gamma)d\mathbb{Q}(\gamma) \sim \mathcal{N}(0, B_\tau^0)$ where B_τ^0 is positive definite by Assumption 5. This fact further implies that

$$\sqrt{n}(\widehat{\theta}_{\tau n} - \theta_\tau^0) \Rightarrow -A_\tau^{0\,-1}\int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma, \theta_\tau^0)\mathcal{G}_\tau(\gamma)d\mathbb{Q}(\gamma) \sim \mathcal{N}(0, C_\tau^0),$$

as required. ■

Proof of Theorem 3: The derivation of (4) is almost identical to that of (3) and is not repeated for brevity. Instead, we focus on deriving the limit distribution from (4).

If we apply (4) to the misspecified model, it now follows that $\sqrt{n}(\widetilde{\theta}_{\tau n} - \theta_\tau^*) = -A_\tau^{*-1}n^{-1/2}\sum_{i=1}^n \widehat{J}_{\tau i} + o_{\mathbb{P}}(1)$. We focus on $n^{-1/2}\sum_{i=1}^n \widehat{J}_{\tau i}$ to derive the limit distribution. Apply (A.1), as given in the proof of

Lemma 3, to the misspecified model giving, for each $\gamma \in \Gamma$,

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{J}_{\tau i} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\mathbb{1}\{\widehat{G}_i(\gamma) \le \rho_\tau(\gamma,\theta_\tau^*)\} - \tau\Big)\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) + \nabla_\pi'\widetilde{G}_i(\gamma,\bar{\pi}_{\gamma n})(\widehat{\pi}_n - \pi^*) \le \rho_\tau(\gamma,\theta_\tau^*)\} - \tau\Big)\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big[\Big(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le \rho_\tau(\gamma,\theta_\tau^*)\} - \tau\Big) - f_\gamma(x_\tau(\gamma))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)]P^{*-1}S_i\Big] + o_{\mathbb{P}}(1).\end{aligned}$$

Here, we applied the ULLN to obtain $n^{-1}\sum_{i=1}^{n}\nabla_\pi\widetilde{G}_i(\cdot,\bar{\pi}_{\gamma n}) \stackrel{\mathbb{P}}{\to} \mathbb{E}[\nabla_\pi\widetilde{G}_i(\cdot,\pi^*)]$ by using Assumption 8. It now follows that

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{J}_{\tau i} =& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^*)\Big(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le \rho_\tau(\gamma,\theta_\tau^*)\} - \tau\Big)\,d\mathbb{Q}(\gamma)\\ &- \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^*)f_\gamma(x_\tau(\gamma))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)]d\mathbb{Q}(\gamma)P^{*-1}S_i + o_{\mathbb{P}}(1).\end{aligned}$$

Here, Assumptions 6 and 8 imply that $\int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^*)f_\gamma(x_\tau(\gamma))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)]d\mathbb{Q}(\gamma)$ is well defined. We further note that $\int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^*)(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le \rho_\tau(\gamma,\theta_\tau^*)\} - \tau)$ and $\int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^*)f_\gamma(x_\tau(\gamma))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)]d\mathbb{Q}(\gamma)$ are defined as $J_{\tau i}$ and K_τ^*, respectively, so that we can rewrite this equation as

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widehat{J}_{\tau i} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(J_{\tau i} - K_\tau^* P^{*-1}S_i) + o_{\mathbb{P}}(1).$$

Given this result, Assumptions 2 and 7 imply that $\mathbb{E}[J_{\tau i}] = 0$ and $\mathbb{E}[S_i] = 0$. Furthermore, Assumption 9 implies that $\widetilde{B}_\tau^* := \mathbb{E}[(J_{\tau i} - K_\tau^* P^{*-1}S_i)(J_{\tau i} - K_\tau^* P^{*-1}S_i)']$ is positive definite, and for each $j = 1, 2, \ldots, c_\tau$, $\mathbb{E}[J_{\tau ij}^2] < \infty$ and $\mathbb{E}[S_{ij}^2] < \infty$ by Assumptions 7 and 8. It now follows by the multivariate CLT that $\sqrt{n}(\widetilde{\theta}_{\tau n} - \theta_\tau^*) \stackrel{\text{A}}{\sim} \mathcal{N}(0, \widetilde{C}_\tau^*)$, as required. ■

Proof of Lemma 3: We first derive the covariance kernel of $\widetilde{\mathcal{G}}_\tau(\cdot)$. Note that for any c, if $a > 0$, $\mathbb{1}\{x \le c - a\} = \mathbb{1}\{x \le c\} - \mathbb{1}\{x \in (c-a, c]\}$. On the other hand, if $a < 0$, $\mathbb{1}\{x \le c - a\} = \mathbb{1}\{x \le c\} + \mathbb{1}\{x \in (c, c-a]\}$. Therefore, $\mathbb{1}\{x \le c - a\} = \mathbb{1}\{x \le c\} - \mathbb{1}\{c - a < x \le c\} + \mathbb{1}\{c < x \le c - a\}$.

We use this equality to show the given claim. For notational simplicity, let $x_\tau(\gamma)$ and $\widehat{\mu}_{ni}(\gamma)$ denote $\rho_\tau(\gamma,\theta_\tau^0)$ and $+\nabla_\pi'\widetilde{G}_i(\gamma,\bar{\pi}_{\gamma n})(\widehat{\pi}_n - \pi^*)$, respectively. If we further let x, c, and a be $\widehat{G}_i(\gamma)$, $x_\tau(\gamma)$, and $\widehat{\mu}_{ni}(\gamma)$, respectively, it now follows that $\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) + \nabla_\pi'\widetilde{G}_i(\gamma,\bar{\pi}_{\gamma n})(\widehat{\pi}_n - \pi^*) \le \rho_\tau(\gamma,\theta_\tau^0)\} = \mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} + \mathbb{1}\{x_\tau(\gamma) < \widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma) - \widehat{\mu}_{ni}(\gamma)\} - \mathbb{1}\{x_\tau(\gamma) - \widehat{\mu}_{ni}(\gamma) < \widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\}$. Note that Assumption 8 implies that $\nabla_\pi\widetilde{G}_i(\cdot,\cdot) = O_{\mathbb{P}}(1)$ and $(\widehat{\pi}_n - \pi^*) = o_{\mathbb{P}}(1)$, so that

$\widehat{\mu}_{ni}(\gamma) = o_{\mathbb{P}}(1)$ uniformly in γ. Therefore,

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \in (x_\tau(\gamma), x_\tau(\gamma)-\widehat{\mu}_{ni}(\gamma)]\} - \mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \in (x_\tau(\gamma)-\widehat{\mu}_{ni}(\gamma), x_\tau(\gamma)]\} \\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)-\widehat{\mu}_{ni}(\gamma)\} - F_\gamma(x_\tau(\gamma))\Big) - \Big(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - F_\gamma(x_\tau(\gamma))\Big) \\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - F_\gamma(x_\tau(\gamma))\Big) - \frac{1}{n}\sum_{i=1}^{n} f_\gamma(x_\tau(\gamma))\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)\sqrt{n}(\widehat{\pi}_n-\pi^*) \\
&\quad - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - F_\gamma(x_\tau(\gamma))\Big) + o_{\mathbb{P}}(1) \\
&= - f_\gamma(x_\tau(\gamma))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)]\sqrt{n}(\widehat{\pi}_n-\pi^*) + o_{\mathbb{P}}(1),
\end{aligned}$$

where the second equality follows from that $n^{-1/2}\sum_{i=1}^{n}\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)-\widehat{\mu}_{ni}(\gamma)\} = n^{-1/2}\sum_{i=1}^{n}\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - n^{-1}\sum_{i=1}^{n}F_\gamma'(x_\tau(\gamma))\sqrt{n}\widehat{\mu}_{ni}(\gamma) + o_{\mathbb{P}}(1)$ and applying the mean-value theorem at the limit. Note that $F_\gamma'(x_\tau(\gamma)) = f_\gamma(x_\tau(\gamma))$. Therefore,

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) + \nabla_\pi'\widetilde{G}_i(\gamma,\bar{\pi}_{\gamma n})(\widehat{\pi}_n-\pi^*) \le \rho_\tau(\gamma,\theta_\tau^0)\} - \tau\Big) \\
&\qquad = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - \tau\Big) - f_\gamma(x_\tau(\gamma))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)]\sqrt{n}(\widehat{\pi}_n-\pi^*) + o_{\mathbb{P}}(1) \\
&\qquad = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big[\Big(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - \tau\Big) - f_\gamma(x_\tau(\gamma))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)]P^{*-1}S_i\Big] + o_{\mathbb{P}}(1) \quad \text{(A.1)}
\end{aligned}$$

Given this, we compute the covariance kernel using the summand on the right side of (A.1), viz.,

$$\begin{aligned}
&\mathbb{E}[[(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - \tau) - f_\gamma(x_\tau(\gamma))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)]P^{*-1}S_i] \\
&\quad \times [(\mathbb{1}\{\widetilde{G}_i(\gamma',\pi^*) \le x_\tau(\gamma')\} - \tau) - f_{\gamma'}(x_\tau(\gamma'))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma',\pi^*)]P^{*-1}S_i]] \\
&\qquad = \mathbb{E}[(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - \tau)(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - \tau)] \\
&\qquad\quad - f_\gamma(x_\tau(\gamma))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)]P^{*-1}\mathbb{E}[S_i(\mathbb{1}\{\widetilde{G}_i(\gamma',\pi^*) \le x_\tau(\gamma')\} - \tau)] \\
&\qquad\quad - f_{\gamma'}(x_\tau(\gamma'))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma',\pi^*)]P^{*-1}\mathbb{E}[S_i(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - \tau)] \\
&\qquad\quad + f_\gamma(x_\tau(\gamma))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)]P^{*-1}H^*P^{*-1}\mathbb{E}[\nabla_\pi\widetilde{G}_i(\gamma',\pi^*)]f_{\gamma'}(x_\tau(\gamma)).
\end{aligned}$$

Observing that $\mathbb{E}[(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - \tau)(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*) \le x_\tau(\gamma)\} - \tau)] = \kappa_\tau(\gamma,\gamma')$, the desired covariance kernel $\widetilde{\kappa}_\tau(\gamma,\gamma')$ is now obtained from this equality.

We next prove that the left side of (A.1) is stochastically equicontinuous. We let $\varsigma(\gamma) := f_\gamma(x_\tau(\gamma))\mathbb{E}[\nabla_\pi'\widetilde{G}_i(\gamma,\pi^*)]P^{*-1}$ for notational simplicity and show that the right side of (A.1) satisfies the bound condition to apply Ossiander's L^2 entropy condition: for some C and $\nu > 0$,

$$\mathbb{E}\left(\sup_{\|\gamma-\gamma'\|<\delta}|(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*)\le x_\tau(\gamma)\}-\varsigma(\gamma)S_i)-(\mathbb{1}\{\widetilde{G}_i(\gamma',\pi^*)\le x_\tau(\gamma')\}-\varsigma(\gamma')S_i)|^2\right)^{1/2}\le C\delta^{\nu}. \tag{A.2}$$

We here note that

$$\begin{aligned}\sup_{\|\gamma-\gamma'\|<\delta}&|(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*)\le x_\tau(\gamma)\}-\varsigma(\gamma)S_i)-(\mathbb{1}\{\widetilde{G}_i(\gamma',\pi^*)\le x_\tau(\gamma')\}-\varsigma(\gamma')S_i)|^2\\ &\le \sup_{\|\gamma-\gamma'\|<\delta}|\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*)\le x_\tau(\gamma)\}-\mathbb{1}\{\widetilde{G}_i(\gamma',\pi^*)\le x_\tau(\gamma')\}|^2\\ &\quad+\sup_{\|\gamma-\gamma'\|<\delta}\|\varsigma(\gamma)-\varsigma(\gamma')\|\cdot\|S_i\|+\sup_{\|\gamma-\gamma'\|<\delta}|(\varsigma(\gamma)-\varsigma(\gamma'))S_i)|^2.\end{aligned}$$

In the proof of Lemma 2, we already saw that there are C_1 and $\nu_1 > 0$ such that

$$\mathbb{E}\left(\sup_{\|\gamma-\gamma'\|<\delta}|(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*)\le x_\tau(\gamma)\})-(\mathbb{1}\{\widetilde{G}_i(\gamma',\pi^*)\le x_\tau(\gamma')\})|^2\right)^{1/2}\le C_1\delta^{\nu_1}.$$

Next, Assumptions 2, 6, and 8 imply that $\varsigma(\cdot)$ is Lipschitz continuous, because the product of two Lipschitz continuous functions is Lipschitz continuous: for some $m > 0$, $\|\varsigma(\gamma)-\varsigma(\gamma')\|\le m\|\gamma-\gamma'\|$, so that for some C_2 and $\nu_2 > 0$,

$$\mathbb{E}\left(\sup_{\|\gamma-\gamma'\|<\delta}|(\varsigma(\gamma)-\varsigma(\gamma'))|\cdot|S_i|\right)\le C_2\delta^{\nu_2}$$

by letting $C_2 := ms\cdot\max_{j=1,\dots,s}\mathbb{E}[|S_{ij}|^2]$ and $\nu_2 = 1$. Note that $\max_{j=1,\dots,s}\mathbb{E}[|S_{ij}|^2]<\infty$ from Assumption 7. We note that

$$|(\varsigma(\gamma)-\varsigma(\gamma'))S_i)|^2\le\max_{j=1,\dots,s}\mathbb{E}[S_{ij}|^2]\cdot\|\varsigma(\gamma)-\varsigma(\gamma')\|^2\le\max_{j=1,\dots,s}\mathbb{E}[S_{ij}|^2]\cdot m^2\|\gamma-\gamma'\|^2,$$

so that if we let $C_3 := m^2\cdot\max_{j=1,\dots,s}\mathbb{E}[|S_{ij}|^2]$ and $\nu_3 = 2$,

$$\mathbb{E}\left(\sup_{\|\gamma-\gamma'\|<\delta}|(\varsigma(\gamma)-\varsigma(\gamma'))S_i|^2\right)\le C_3\delta^{\nu_3}.$$

Therefore, if we let $C := \max[C_1, C_2, C_3]$ and $\nu := \max[\nu_1, \nu_2, \nu_3]$, the desired inequality in (A.2) follows. This shows that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\mathbb{1}\{\widetilde{G}_i(\gamma,\pi^*)+\nabla'_\pi\widetilde{G}_i(\gamma,\bar{\pi}_{\gamma n})(\widehat{\pi}_n-\pi^*)\le\rho_\tau(\gamma,\theta^0_\tau)\}-\tau\right)$$

is stochastically equicontinuous, completing the proof. ■

Proof of Theorem 4: Given Lemma 3, we note that

$$\int_{\gamma} \nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^0)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\mathbb{1}\{\widehat{G}_i(\gamma)\le \rho_\tau(\gamma,\theta_\tau^0)\}-\tau\right)d\mathbb{Q}(\gamma) \Rightarrow \int_{\gamma}\nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^0)\widetilde{\mathcal{G}}_\tau(\gamma)d\mathbb{Q}(\gamma)$$

by applying the continuous mapping theorem. The final integral follows a normal distribution from the fact that $\widetilde{\mathcal{G}}_\tau(\cdot)$ is a Gaussian stochastic process. By dominated convergence using Assumption 14,

$$\mathbb{E}\left[\int_{\gamma}\nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^0)\widetilde{\mathcal{G}}_\tau(\gamma)d\mathbb{Q}(\gamma)\right] = \int_{\gamma}\nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^0)\mathbb{E}[\widetilde{\mathcal{G}}_\tau(\gamma)]d\mathbb{Q}(\gamma)=0 \quad \text{and}$$

$$\begin{aligned}\mathbb{E}&\left[\int_{\gamma}\int_{\gamma'}\nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^0)\widetilde{\mathcal{G}}_\tau(\gamma)\widetilde{\mathcal{G}}_\tau(\gamma')\nabla_{\theta_\tau}\rho_\tau(\gamma',\theta_\tau^0)d\mathbb{Q}(\gamma)d\mathbb{Q}(\gamma')\right]\\ &= \int_{\gamma}\int_{\gamma'}\nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^0)\mathbb{E}[\widetilde{\mathcal{G}}_\tau(\gamma)\widetilde{\mathcal{G}}_\tau(\gamma')]\nabla'_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^0)d\mathbb{Q}(\gamma)d\mathbb{Q}(\gamma')\\ &= \int_{\gamma}\int_{\gamma'}\nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^0)\widetilde{\kappa}_\tau(\gamma,\gamma')\nabla'_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^0)d\mathbb{Q}(\gamma)d\mathbb{Q}(\gamma')\end{aligned}$$

which is defined as $\widetilde{B}_\tau^0$. Therefore, $\int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^0)\widetilde{\mathcal{G}}_\tau(\gamma)d\mathbb{Q}(\gamma)\sim\mathcal{N}(0,\widetilde{B}_\tau^0)$, implying that

$$\sqrt{n}(\widetilde{\theta}_{\tau n}-\theta_\tau^0)\Rightarrow -A_\tau^{0\,-1}\int_{\gamma}\nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^0)\widetilde{\mathcal{G}}_\tau(\gamma)d\mathbb{Q}(\gamma)\sim\mathcal{N}(0,\widetilde{C}_\tau^0),$$

giving the desired result. ■

Proof of Theorem 5: (*i*) As the proof of the consistency in (*i*) is almost identical to that of (*ii*), we prove only (*i*).

(*i.a*) If we apply the mean-value theorem to $\rho_\tau(\gamma,\widehat{\theta}_{\tau n})$ and $\nabla_{\theta_\tau}\rho_\tau(\gamma,\widehat{\theta}_{\tau n})$ around the unknown parameter θ_τ^*, for each γ, there are $\bar{\theta}_{\tau\gamma}^*$ and $\acute{\theta}_{\tau\gamma}^*$ such that

$$\rho_\tau(\gamma,\widehat{\theta}_{\tau n})=\rho_\tau(\gamma,\theta_\tau^*)+\nabla'_{\theta_\tau}\rho_\tau(\gamma,\bar{\theta}_{\tau\gamma}^*)(\widehat{\theta}_{\tau n}-\theta_\tau^*) \quad \text{and}$$

$$\nabla_{\theta_\tau}\rho_\tau(\gamma,\widehat{\theta}_{\tau n})=\nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^*)+\nabla_\theta^2\rho_\tau(\gamma,\acute{\theta}_{\tau\gamma}^*)(\widehat{\theta}_{\tau n}-\theta_\tau^*).$$

For notational simplicity, let $\widehat{\nu}_n(\gamma):=-\nabla'_{\theta_\tau}\rho_\tau(\gamma,\bar{\theta}_{\tau\gamma}^*)(\widehat{\theta}_{\tau n}-\theta_\tau^*)$. Given these expressions, note that

$$\begin{aligned}\mathbb{1}\{G_i(\gamma)\le\rho_\tau(\gamma,\widehat{\theta}_{\tau n})\}-\tau &= \mathbb{1}\{G_i(\gamma)\le\rho_\tau(\gamma,\theta_\tau^*)\}-\tau\\ &+\mathbb{1}\{\rho_\tau(\gamma,\theta_\tau^*)<G_i(\gamma)\le\rho_\tau(\gamma,\theta_\tau^*)-\widehat{\nu}_n(\gamma)\}-\mathbb{1}\{\rho_\tau(\gamma,\theta_\tau^*)-\widehat{\nu}_n(\gamma)<G_i(\gamma)\le\rho_\tau(\gamma,\theta_\tau^*)\}\\ &=\mathbb{1}\{G_i(\gamma)\le\rho_\tau(\gamma,\theta_\tau^*)\}-\tau+o_{\mathbb{P}}(1)\end{aligned} \tag{A.3}$$

using the fact that $\mathbb{1}\{\rho_\tau(\gamma,\theta_\tau^*)<G_i(\gamma)\le\rho_\tau(\gamma,\theta_\tau^*)-\widehat{\nu}_n(\gamma)\}=o_{\mathbb{P}}(1)$ and $\mathbb{1}\{\rho_\tau(\gamma,\theta_\tau^*)-\widehat{\nu}_n(\gamma)<G_i(\gamma)\le$

$\rho_\tau(\gamma,\theta_\tau^*)\} = o_{\mathbb{P}}(1)$ from the fact that for each γ, $\widehat{\nu}_n(\gamma) = o_{\mathbb{P}}(1)$. It follows that

$$\begin{aligned} J_{\tau ni} &:= \int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma,\widehat{\theta}_{\tau n})\left(\mathbb{1}\{G_i(\gamma)\le \rho_\tau(\gamma,\widehat{\theta}_{\tau n})\} - \tau\right) d\mathbb{Q}(\gamma) \\ &= \int_\gamma \nabla_{\theta_\tau}\rho_\tau(\gamma,\theta_\tau^*)\left(\mathbb{1}\{G_i(\gamma)\le \rho_\tau(\gamma,\theta_\tau^*)\} - \tau\right) d\mathbb{Q}(\gamma) + o_{\mathbb{P}}(1) = J_{\tau i} + o_{\mathbb{P}}(1), \end{aligned}$$

given that for each j and $j' = 1,2,\ldots,c_\tau$, $|\partial^2/(\partial\theta_{\tau j}\partial\theta_{\tau j'})\rho_\tau(\cdot,\cdot)| \le M < \infty$ and $|\partial/(\partial\theta_{\tau j})\rho_\tau(\cdot,\cdot)| \le M < \infty$ from Assumption 3. Thus,

$$\widehat{B}_{\tau n} := \frac{1}{n}\sum_{i=1}^n J_{\tau ni}J_{\tau ni}' = \frac{1}{n}\sum_{i=1}^n J_{\tau i}J_{\tau i}' + o_{\mathbb{P}}(1).$$

We now further note that for each $j = 1,2,\ldots,c_\tau$, $\mathbb{E}[J_{\tau ij}^2] < \infty$ from Assumption 3, so that it now follows that $\widehat{B}_{\tau n} = \frac{1}{n}\sum_{i=1}^n J_{\tau i}J_{\tau i}' + o_{\mathbb{P}}(1) \overset{\mathbb{P}}{\to} \mathbb{E}[J_{\tau i}J_{\tau i}'] =: B_\tau^*$. Therefore, $\widehat{B}_{\tau n} \overset{\mathbb{P}}{\to} B_\tau^*$.

Proof of consistency of $\widetilde{B}_{\tau n}$ is not detailed because it follows in a similar fashion to the consistency of $\widehat{B}_{\tau n}$. In particular, given the moment conditions in Assumption 8 and the condition for the other consistent estimators for P^*, H^*, and K_τ^* as given in Assumption 11, it follows that $\widehat{J}_{\tau ni} = \widehat{J}_{\tau i} + o_{\mathbb{P}}(1)$, and the result $\widetilde{B}_{\tau n} \overset{\mathbb{P}}{\to} \widetilde{B}_\tau^*$ follows.

(*i.b*) Given the second-order differentiability of $\rho_\tau(\gamma,\cdot)$ in Assumption 2 and Theorem 3(*i*), we apply (A.3) to obtain that

$$\mathbb{1}\{G_i(\gamma)\le \rho_\tau(\gamma,\widehat{\theta}_{\tau n})\} - \tau = \mathbb{1}\{G_i(\gamma)\le \rho_\tau(\gamma,\theta_\tau^0)\} - \tau + o_{\mathbb{P}}(1),$$

implying that $\widehat{\kappa}_{\tau n}(\gamma,\gamma') = \widehat{\kappa}_\tau(\gamma,\gamma') + o_{\mathbb{P}}(1)$, where

$$\widehat{\kappa}_\tau(\gamma,\gamma') := \frac{1}{n}\sum_{i=1}^n(\mathbb{1}\{G_i(\gamma)\le \rho_\tau(\gamma,\theta_\tau^0)\} - \tau)(\mathbb{1}\{G_i(\gamma)\le \rho_\tau(\gamma,\theta_\tau^0)\} - \tau).$$

Furthermore, $\nabla_{\theta_\tau}\rho_\tau(\cdot,\widehat{\theta}_{\tau n}) \overset{\mathbb{P}}{\to} \nabla_{\theta_\tau}\rho_\tau(\cdot,\theta_\tau^0)$ from the fact that $\widehat{\theta}_{\tau n} \overset{\mathbb{P}}{\to} \theta_\tau^0$ and the continuity of $\rho_\tau(\cdot,\cdot)$. Therefore, from the definition of $\widehat{B}_{\tau n}^\sharp$, if $\widehat{\kappa}_\tau(\cdot,\cdot)$ is consistent for $\kappa_\tau(\cdot,\cdot)$ uniformly on $\Gamma\times\Gamma$, then the desired result follows.

For the proof of the consistency of $\widehat{\kappa}_\tau(\cdot,\cdot)$, we note that

$$\begin{aligned} \widehat{\kappa}_\tau(\gamma,\gamma') :=& \frac{1}{n}\sum_{i=1}^n \mathbb{1}\{G_i(\gamma)\le \rho_\tau(\gamma,\theta_\tau^0)\}\mathbb{1}\{G_i(\gamma')\le \rho_\tau(\gamma',\theta_\tau^0)\} \\ &- \frac{\tau}{n}\sum_{i=1}^n \mathbb{1}\{G_i(\gamma)\le \rho_\tau(\gamma,\theta_\tau^0)\} - \frac{\tau}{n}\sum_{i=1}^n \mathbb{1}\{G_i(\gamma')\le \rho_\tau(\gamma',\theta_\tau^0)\} + \tau^2. \end{aligned}$$

Here, the uniform consistency of $n^{-1}\sum_{i=1}^n \mathbb{1}\{G_i(\cdot)\le \rho_\tau(\cdot,\theta_\tau^0)\}$ follows if $\{n^{-1}\sum_{i=1}^n \mathbb{1}\{G_i(\cdot)\le \rho_\tau(\cdot,\theta_\tau^0)\}\}$ is stochastically equicontinuous as shown in Newey (1991). Note that the proof of Lemma 2 already shows that $\{n^{-1}\sum_{i=1}^n \mathbb{1}\{G_i(\cdot) \le \rho_\tau(\cdot, \theta_\tau^0)\}\}$ is stochastically equicontinuous.

We therefore only show that $\{n^{-1}\sum_{i=1}^n \mathbb{1}\{G_i(\cdot) \le \rho_\tau(\cdot, \theta_\tau^0)\}\mathbb{1}\{G_i(\cdot') \le \rho_\tau(\cdot', \theta_\tau^0)\}\}$ is stochastically equicontinuous for the uniform continuity of $\widehat{\kappa}_\tau(\cdot,\cdot)$, where "$(\cdot')$" is used to distinguish it from "$(\cdot)$". For this purpose, we use Ossiander's L^2 entropy condition as in the proof of Lemma 2: if we let $U_i(\gamma) := F_\gamma(G_i(\gamma))$ be the PIT of $G_i(\gamma)$ as in the proof of Lemma 2,

$$\begin{aligned}&\left|\mathbb{1}\{G_i(\gamma) \le \rho_\tau(\gamma, \theta_\tau^0)\}\mathbb{1}\{G_i(\gamma') \le \rho_\tau(\gamma', \theta_\tau^0)\} - \mathbb{1}\{G_i(\gamma'') \le \rho_\tau(\gamma'', \theta_\tau^0)\}\mathbb{1}\{G_i(\gamma''') \le \rho_\tau(\gamma''', \theta_\tau^0)\}\right| \\ &\quad = \left|\mathbb{1}\{U_i(\gamma) \le \tau\}\mathbb{1}\{U_i(\gamma') \le \tau\} - \mathbb{1}\{U_i(\gamma'') \le \tau\}\mathbb{1}\{U_i(\gamma''') \le \tau\}\right|,\end{aligned}$$

so that Ossiander's L^2 entropy condition requires that there are are $\nu > 0$ and $C > 0$ such that

$$\mathbb{E}\left[\sup_{\|(\gamma,\gamma')-(\gamma'',\gamma''')\|<\delta}\left|\mathbb{1}\{U_i(\gamma) \le \tau\}\mathbb{1}\{U_i(\gamma') \le \tau\} - \mathbb{1}\{U_i(\gamma'') \le \tau\}\mathbb{1}\{U_i(\gamma''') \le \tau\}\right|^2\right] \le C^0\delta^\nu. \tag{A.4}$$

We first note that

$$\begin{aligned}&\left|\mathbb{1}\{U_i(\gamma) \le \tau\}\mathbb{1}\{U_i(\gamma') \le \tau\} - \mathbb{1}\{U_i(\gamma'') \le \tau\}\mathbb{1}\{U_i(\gamma''') \le \tau\}\right| \\ &\qquad = |\mathbb{1}\{U_i(\gamma) \le \tau\}(\mathbb{1}\{U_i(\gamma') \le \tau\} - \mathbb{1}\{U_i(\gamma'') \le \tau\}) \\ &\qquad\qquad + \mathbb{1}\{U_i(\gamma'') \le \tau\}(\mathbb{1}\{U_i(\gamma) \le \tau\} - \mathbb{1}\{U_i(\gamma''') \le \tau\})| \\ &\qquad \le \left|\mathbb{1}\{U_i(\gamma') \le \tau\} - \mathbb{1}\{U_i(\gamma'') \le \tau\}\right| + \left|\mathbb{1}\{U_i(\gamma) \le \tau\} - \mathbb{1}\{U_i(\gamma''') \le \tau\}\right|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}\left[\sup_{\|(\gamma,\gamma')-(\gamma'',\gamma''')\|<\delta}\left|\mathbb{1}\{U_i(\gamma) \le \tau\}\mathbb{1}\{U_i(\gamma') \le \tau\} - \mathbb{1}\{U_i(\gamma'') \le \tau\}\mathbb{1}\{U_i(\gamma''') \le \tau\}\right|^2\right] \\ &\quad \le 2\mathbb{E}\left[\sup_{\|\gamma-\gamma''\|<\delta}\left|\mathbb{1}\{U_i(\gamma) \le \tau\} - \mathbb{1}\{U_i(\gamma'') \le \tau\}\right|^2\right] \\ &\quad + 2\mathbb{E}\left[\sup_{\|\gamma-\gamma'\|<\delta}\left|\mathbb{1}\{U_i(\gamma) \le \tau\} - \mathbb{1}\{U_i(\gamma') \le \tau\}\right| \cdot \sup_{\|\gamma''-\gamma'''\|<\delta}\left|\mathbb{1}\{U_i(\gamma'') \le \tau\} - \mathbb{1}\{U_i(\gamma''') \le \tau\}\right|\right].\end{aligned}$$

We here note that

$$\begin{aligned}&\mathbb{E}\left[\sup_{\|\gamma-\gamma'\|<\delta}\left|\mathbb{1}\{U_i(\gamma) \le \tau\} - \mathbb{1}\{U_i(\gamma') \le \tau\}\right| \cdot \sup_{\|\gamma''-\gamma'''\|<\delta}\left|\mathbb{1}\{U_i(\gamma'') \le \tau\} - \mathbb{1}\{U_i(\gamma''') \le \tau\}\right|\right]^2 \\ &\qquad \le \mathbb{E}\left[\left(\sup_{\|\gamma-\gamma'\|<\delta}\left|\mathbb{1}\{U_i(\gamma) \le \tau\} - \mathbb{1}\{U_i(\gamma') \le \tau\}\right|\right)^2\right] \\ &\qquad\qquad \times \mathbb{E}\left[\left(\sup_{\|\gamma''-\gamma'''\|<\delta}\left|\mathbb{1}\{U_i(\gamma'') \le \tau\} - \mathbb{1}\{U_i(\gamma''') \le \tau\}\right|\right)^2\right]\end{aligned}$$

by applying Cauchy-Schwarz. Note that

$$\mathbb{E}\left[\left\{\sup_{\|\gamma''-\gamma'''\|<\delta}\left|\mathbb{1}\{U_i(\gamma'')\le\tau\}-\mathbb{1}\{U_i(\gamma''')\le\tau\}\right|\right\}^2\right]$$
$$\le \mathbb{E}\left[\sup_{\|\gamma''-\gamma'''\|<\delta}\left|\mathbb{1}\{U_i(\gamma'')\le\tau\}-\mathbb{1}\{U_i(\gamma''')\le\tau\}\right|^2\right].$$

It follows that

$$\mathbb{E}\left[\sup_{\|\gamma-\gamma'\|<\delta}\left|\mathbb{1}\{U_i(\gamma)\le\tau\}-\mathbb{1}\{U_i(\gamma')\le\tau\}\right|\cdot\sup_{\|\gamma''-\gamma'''\|<\delta}\left|\mathbb{1}\{U_i(\gamma'')\le\tau\}-\mathbb{1}\{U_i(\gamma''')\le\tau\}\right|\right]^2$$
$$\le \mathbb{E}\left[\sup_{\|\gamma''-\gamma'''\|<\delta}\left|\mathbb{1}\{U_i(\gamma'')\le\tau\}-\mathbb{1}\{U_i(\gamma''')\le\tau\}\right|^2\right]^2,$$

implying that

$$\mathbb{E}\left[\sup_{\|(\gamma,\gamma')-(\gamma'',\gamma''')\|<\delta}\left|\mathbb{1}\{U_i(\gamma)\le\tau\}\mathbb{1}\{U_i(\gamma')\le\tau\}-\mathbb{1}\{U_i(\gamma'')\le\tau\}\mathbb{1}\{U_i(\gamma''')\le\tau\}\right|^2\right]$$
$$\le 2\mathbb{E}\left[\sup_{\|\gamma-\gamma''\|<\delta}\left|\mathbb{1}\{U_i(\gamma)\le\tau\}-\mathbb{1}\{U_i(\gamma'')\le\tau\}\right|^2\right]$$
$$+2\mathbb{E}\left[\sup_{\|\gamma''-\gamma'''\|<\delta}\left|\mathbb{1}\{U_i(\gamma'')\le\tau\}-\mathbb{1}\{U_i(\gamma''')\le\tau\}\right|^2\right]$$
$$=4\mathbb{E}\left[\sup_{\|\gamma''-\gamma'''\|<\delta}\left|\mathbb{1}\{U_i(\gamma'')\le\tau\}-\mathbb{1}\{U_i(\gamma''')\le\tau\}\right|^2\right].$$

We have already seen in the proof of Lemma 2 that there are $\nu>0$ and $C>0$ such that

$$\mathbb{E}\left[\sup_{\|\gamma''-\gamma'''\|<\delta}\left|\mathbb{1}\{U_i(\gamma'')\le\tau\}-\mathbb{1}\{U_i(\gamma''')\le\tau\}\right|^2\right]\le C\delta^\nu,$$

so that

$$\mathbb{E}\left[\sup_{\|(\gamma,\gamma')-(\gamma'',\gamma''')\|<\delta}\left|\mathbb{1}\{U_i(\gamma)\le\tau\}\mathbb{1}\{U_i(\gamma')\le\tau\}-\mathbb{1}\{U_i(\gamma'')\le\tau\}\mathbb{1}\{U_i(\gamma''')\le\tau\}\right|^2\right]\le 4C\delta^\nu.$$

We now let $C^0:=4C$ in (A.4) for the same ν to complete the proof that $\{n^{-1}\sum_{i=1}^n\mathbb{1}\{G_i(\cdot)\le\rho_\tau(\cdot,\theta_\tau^0)\}\mathbb{1}\{G_i(\cdot')\le\rho_\tau(\cdot',\theta_\tau^0)\}\}$ is stochastically equicontinuous. This completes the proof. ■

Proof of Theorem 6: Theorem 3 (*ii*) implies that for each $j=1,2,\ldots,p$, $n^{-1/2}\sum_{i=1}^n\widehat{J}_{\tau_j i}\overset{\text{A}}{\sim}\mathcal{N}(0,\widetilde{B}^*_{\tau_j})$. In addition, $\widetilde{B}^*_\tau$ is positive definite by Assumption 15. It therefore follows by the Cramér-Wold device that $n^{-1/2}\sum_{i=1}^n\widehat{J}_i\overset{\text{A}}{\sim}\mathcal{N}(0,\widetilde{B}^*)$, so that $\sqrt{n}(\widetilde{\theta}_n-\theta^*)=-A^{*-1}n^{-1/2}\sum_{i=1}^n\widehat{J}_i+o_{\mathbb{P}}(1)\overset{\text{A}}{\sim}\mathcal{N}(0,\widetilde{C}^*)$. ■

Proof of Lemma 4: To show the claim, for each $j = 1, 2, \ldots, p$, we first let

$$\widehat{\omega}_{nj}(\gamma) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{1}\{\widehat{G}_i(\gamma) \leq \rho_{\tau_j}(\gamma, \theta^0_{\tau_j})\}$$

for notational simplicity. Second, note that Lemma 3 implies that for any $\epsilon_j > 0$ and $\eta_j > 0$, there exist n_{0j} and $\delta_j > 0$ such that if $n > n_{0j}$,

$$\mathbb{P}\left(\sup_{\|\gamma-\gamma'\|<\delta_j} \left|\widehat{\omega}_{nj}(\gamma) - \widehat{\omega}_{nj}(\gamma')\right| > \epsilon_j \right) < \eta_j. \tag{A.5}$$

Third, applying the Cramér-Wold device gives the desired result. That is, for all $\lambda \in \mathbb{R}^p$ such that $\lambda'\lambda = 1$, if we show that for all $\epsilon > 0$ and $\eta > 0$, there are n_0 and $\delta > 0$ such that if $n > n_0$,

$$\mathbb{P}\left(\sup_{\|\gamma-\gamma'\|<\delta} \left| \sum_{j=1}^{p} \lambda_j(\widehat{\omega}_{nj}(\gamma) - \tau_j) - \sum_{j=1}^{p} \lambda_j(\widehat{\omega}_{nj}(\gamma') - \tau_j) \right| > \epsilon \right) < \eta, \tag{A.6}$$

the desired result follows as in Wooldridge and White (1988, proposition 4.1). Here, for each $j = 1, 2, \ldots, p$, λ_j denotes the j-th row element of λ.

To show (A.6), we let $\epsilon > 0$ and $\eta > 0$ and show the stochastic equicontinuity using its definition. If $\lambda_j \neq 0$, we let ϵ_j and η_j be $\epsilon/(p \cdot |\lambda_j|)$ and η/p, respectively. Then, it follows that if $n > n_0 := \max[n_{01}, n_{02}, \ldots, n_{0p}]$,

$$\mathbb{P}\left(\sup_{\|\gamma-\gamma'\|<\delta_j} \left|\widehat{\omega}_{nj}(\gamma) - \widehat{\omega}_{nj}(\gamma')\right| > \frac{\epsilon}{p \cdot |\lambda_j|} \right) < \frac{\eta}{p}$$

from (A.5). On the other hand, if $\lambda_j = 0$,

$$\mathbb{P}\left(\sup_{\|\gamma-\gamma'\|<\delta_j} |\lambda_j| \cdot \left|\widehat{\omega}_{nj}(\gamma) - \widehat{\omega}_{nj}(\gamma')\right| > \frac{\epsilon}{p} \right) = 0 < \frac{\eta}{p}.$$

Therefore,

$$\sum_{j=1}^{p} \mathbb{P}\left(\sup_{\|\gamma-\gamma'\|<\delta_j} |\lambda_j| \cdot \left|\widehat{\omega}_{nj}(\gamma) - \widehat{\omega}_{nj}(\gamma')\right| > \epsilon \right) < \eta,$$

and we also note that

$$\begin{aligned}
\eta &> \sum_{j=1}^{p} \mathbb{P}\left(\sup_{\|\gamma-\gamma'\|<\delta_j} |\lambda_j| \cdot \left|\widehat{\omega}_{nj}(\gamma) - \widehat{\omega}_{nj}(\gamma')\right| > \epsilon \right) \geq \mathbb{P}\left(\sum_{j=1}^{p} \sup_{\|\gamma-\gamma'\|<\delta_j} |\lambda_j| \cdot \left|\widehat{\omega}_{nj}(\gamma) - \widehat{\omega}_{nj}(\gamma')\right| > \epsilon \right) \\
&\geq \mathbb{P}\left(\sum_{j=1}^{p} \sup_{\|\gamma-\gamma'\|<\delta_j} \left|\lambda_j\{\widehat{\omega}_{nj}(\gamma) - \widehat{\omega}_{nj}(\gamma')\}\right| > \epsilon \right) \geq \mathbb{P}\left(\sum_{j=1}^{p} \sup_{\|\gamma-\gamma'\|<\delta} \left|\lambda_j(\widehat{\omega}_{nj}(\gamma) - \widehat{\omega}_{nj}(\gamma'))\right| > \epsilon \right)
\end{aligned}$$

by letting $\delta := \min[\delta_1, \delta_2, \ldots, \delta_p]$. That is, for each $\epsilon > 0$ and $\eta > 0$, there are n_0 and $\delta > 0$ such that

$$\mathbb{P}\left(\sup_{\|\gamma-\gamma'\|<\delta} \left| \sum_{j=1}^{p} \lambda_j(\widehat{\omega}_{nj}(\gamma) - \tau_j) - \sum_{j=1}^{p} \lambda_j(\widehat{\omega}_{nj}(\gamma') - \tau_j) \right| > \epsilon \right) < \eta.$$

This completes the proof. ■

Proof of Theorem 7: Given Lemma 4, note that

$$\int_\gamma \nabla_\theta \rho(\gamma, \theta^0) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\mathbb{1}\{\widehat{G}_i(\gamma) \le \rho(\gamma, \theta^0)\} - \tau \right) d\mathbb{Q}(\gamma) \Rightarrow \int_\gamma \nabla_\theta \rho(\gamma, \theta^0) \widetilde{\mathcal{G}}(\gamma) d\mathbb{Q}(\gamma)$$

by continuous mapping. Further note that the final integral follows a normal distribution from the fact that for each $j = 1, 2, \ldots, p$, $\widetilde{\mathcal{G}}_{\tau_j}(\cdot)$ is a Gaussian stochastic process. By dominated convergence theorem using Assumption 16,

$$\mathbb{E}\left[\int_\gamma \nabla_\theta \rho(\gamma, \theta^0) \widetilde{\mathcal{G}}(\gamma) d\mathbb{Q}(\gamma) \right] = \int_\gamma \nabla_\theta \rho(\gamma, \theta^0) \mathbb{E}[\widetilde{\mathcal{G}}(\gamma)] d\mathbb{Q}(\gamma) = 0, \quad \text{and}$$

defining $\widetilde{\kappa}(\cdot, \cdot) : \Gamma \times \Gamma \mapsto \mathbb{R}^{p \times p}$ such that its j-th row and j'-th column element is $\widetilde{\kappa}_{\tau_j, \tau_{j'}}(\cdot, \cdot)$ given in Lemma 4,

$$\begin{aligned} \mathbb{E}&\left[\int_\gamma \int_{\gamma'} \nabla_\theta \rho(\gamma, \theta^0) \widetilde{\mathcal{G}}(\gamma) \widetilde{\mathcal{G}}(\gamma') \nabla_\theta \rho(\gamma', \theta^0) d\mathbb{Q}(\gamma) d\mathbb{Q}(\gamma') \right] \\ &= \int_\gamma \int_{\gamma'} \nabla_\theta \rho(\gamma, \theta^0) \mathbb{E}[\widetilde{\mathcal{G}}(\gamma) \widetilde{\mathcal{G}}(\gamma')'] \nabla'_\theta \rho(\gamma, \theta^0) d\mathbb{Q}(\gamma) d\mathbb{Q}(\gamma') \\ &= \int_\gamma \int_{\gamma'} \nabla_\theta \rho(\gamma, \theta^0) \widetilde{\kappa}(\gamma, \gamma') \nabla'_\theta \rho(\gamma, \theta^0) d\mathbb{Q}(\gamma) d\mathbb{Q}(\gamma') \end{aligned}$$

which yields $\widetilde{B}^0$. Therefore, $\int_\gamma \nabla_\theta \rho(\gamma, \theta^0) \widetilde{\mathcal{G}}(\gamma) d\mathbb{Q}(\gamma) \sim \mathcal{N}(0, \widetilde{B}^0)$. Given that $\widetilde{B}^0$ is positive definite by Assumption 16, it follows that

$$\sqrt{n}(\widetilde{\theta}_n - \theta^0) \Rightarrow -A^{0\,-1} \int_\gamma \nabla_\theta \rho(\gamma, \theta^0) \widetilde{\mathcal{G}}(\gamma) d\mathbb{Q}(\gamma) \sim \mathcal{N}(0, \widetilde{C}^0),$$

as desired, completing the proof. ■

Proof of Lemma 5: Since the asymptotic approximation of $\bar{\theta}_n$ is the same as that of $\ddot{\theta}_n$, we prove only (*ii*). Given that $\widehat{q}_n(\cdot)$ is stochastically differentiable in the sense of Pollard (1985, theorem 5), we can construct the Lagrange function to obtain the CTSFQR estimator (see also Newey and McFadden, 1994, section 7). The asymptotic first-order conditions are

$$\Omega \ddot{Q}_n + \ddot{D}'_n \ddot{\lambda}_n = o_{\mathbb{P}}(1) \quad \text{and} \quad R(\ddot{\theta}_n) \equiv 0, \tag{A.7}$$

where $\ddot{\lambda}_n$ stands for the asymptotic Lagrange multiplier. Note further that

$$\Omega\ddot{Q}_n = \Omega\widehat{Q}_n + \Omega A^*(\ddot{\theta}_n - \theta^*) + o_{\mathbb{P}}(1) \quad \text{and} \quad R(\ddot{\theta}_n) = R(\theta^*) + D^*(\widetilde{\theta}_n)(\ddot{\theta}_n - \theta^*) + o_{\mathbb{P}}(1), \tag{A.8}$$

where $\widehat{Q}_n := (n^{-1}\sum_{i=1}^n \widehat{J}_i)$. Solving for $(\ddot{\theta}_n - \theta^*)$ from these two conditions, it now follows that

$$\begin{aligned}\sqrt{n}(\ddot{\theta}_n - \theta^*) = ((\Omega A^*)^{-1} + (\Omega A^*)^{-1}D^{*\prime}E^{*-1}D^*(\Omega A^*)^{-1})\sqrt{n}\Omega\ddot{Q}_n \\ + ((\Omega A^*)^{-1}D^{*\prime}E^{*-1})\sqrt{n}R(\theta^*) + o_{\mathbb{P}}(\sqrt{n}),\end{aligned}$$

where $E^* := -D^*(\Omega A^*)^{-1}D^{*\prime}$ and $\sqrt{n}\Omega\widehat{Q}_n \overset{\text{A}}{\sim} \mathcal{N}(0, \Omega\widetilde{B}^*\Omega)$ by applying Theorem 6. Next,

$$\begin{aligned}&((\Omega A^*)^{-1} + (\Omega A^*)^{-1}D^{*\prime}E^{*-1}D^*(\Omega A^*)^{-1})\sqrt{n}\Omega\ddot{Q}_n \\ &\overset{\text{A}}{\sim} \mathcal{N}(0,(\Omega A^*)^{-1}[(\Omega A^*) + D^{*\prime}E^{*-1}D^*](\Omega A^*)^{-1}\Omega\widetilde{B}^*\Omega(\Omega A^*)^{-1}[(\Omega A^*) + D^{*\prime}E^{*-1}D^*](\Omega A^*)^{-1}).\end{aligned}$$

Here, we note that Ω and A^* are block diagonal matrices, so that the given asymptotic variance matrix simplifies to

$$[I + (\Omega A^*)^{-1}D^{*\prime}E^{*-1}D^*]\widetilde{C}^*[I + D^{*\prime}E^{*-1}D^*(\Omega A^*)^{-1}],$$

so that $\sqrt{n}\{(\ddot{\theta}_n - \theta^*) - ((\Omega A^*)^{-1}D^{*\prime}E^{*-1})R(\theta^*)\} \overset{\text{A}}{\sim} \mathcal{N}(0, [I + (\Omega A^*)^{-1}D^{*\prime}E^{*-1}D^*]\widetilde{C}^*[I + D^{*\prime}E^{*-1}D^* (\Omega A^*)^{-1}])$. Substituting $-D^*(\Omega A^*)^{-1}D^{*\prime}$ for E^*, the desired result follows. ■

Proof of Theorem 8: (*ii*) Since the proofs of (*i*) are almost identical to those of (*ii*), we prove only (*ii*).

(*ii.a*) Applying the mean-value theorem, $R(\widetilde{\theta}_n) = R(\theta^*) + \nabla_\theta' R(\theta_n^\flat)(\widetilde{\theta}_n - \theta^*)$ for some $\theta_n^\flat$ between $\widetilde{\theta}_n$ and θ^*, and if $\mathbb{H}_o$ is imposed, $\sqrt{n}R(\widetilde{\theta}_n) = \nabla_\theta' R(\theta_n^\flat)\sqrt{n}(\widetilde{\theta}_n - \theta^*)$. Note that $\theta_n^\flat \overset{\mathbb{P}}{\to} \theta^*$, so that $\sqrt{n}R(\widetilde{\theta}_n) = \nabla_\theta' R(\theta^*)\sqrt{n}(\widetilde{\theta}_n - \theta^*) + o_{\mathbb{P}}(1)$. Therefore, $\sqrt{n}R(\widetilde{\theta}_n) = \nabla_\theta' R(\theta^*)\sqrt{n}(\widetilde{\theta}_n - \theta^*) \overset{\text{A}}{\sim} \mathcal{N}(0, \nabla_\theta' R(\theta^*)\widetilde{C}^*\nabla_\theta R (\theta^*))$ by Theorem 6 (*ii*). Since $\widetilde{D}_n \overset{\mathbb{P}}{\to} \nabla_\theta' R(\theta^*)$ it follows that $\widetilde{D}_n\widetilde{C}_n\widetilde{D}_n'$ consistently estimates the asymptotic variance matrix of $\sqrt{n}R(\widetilde{\theta}_n)$ from the fact that $\widetilde{A}_n$ is consistent for A^*. It therefore follows that $\ddot{\mathcal{W}}_n := nR(\widetilde{\theta}_n)'\{\widetilde{D}_n\widetilde{C}_n\widetilde{D}_n'\}^{-1}R(\widetilde{\theta}_n) \overset{\text{A}}{\sim} \mathcal{X}_r^2$ under $\mathbb{H}_o$.

Under $\mathbb{H}_a$, $\sqrt{n}R(\widetilde{\theta}_n) = \sqrt{n}R(\theta^*) + \nabla_\theta' R(\theta_n^\flat)\sqrt{n}(\widetilde{\theta}_n - \theta^*)$, so that $\sqrt{n}R(\widetilde{\theta}_n) = O_{\mathbb{P}}(\sqrt{n})$ because $\sqrt{n}R(\theta^*) = O(\sqrt{n})$ and $\nabla_\theta' R(\theta_n^\flat)\sqrt{n}(\widetilde{\theta}_n - \theta^*) = O_{\mathbb{P}}(1)$, implying that $\ddot{\mathcal{W}}_n = O_{\mathbb{P}}(n)$. Therefore, if $c_n = o(n)$, $\lim_{n\to\infty} \mathbb{P}(\ddot{\mathcal{W}} \geq c_n) = 1$.

(*ii.b*) Solving for $\ddot{\lambda}_n$ from (A.7) and (A.8), $\sqrt{n}\ddot{\lambda}_n = -(E^{*-1}D^*(\Omega A^*)^{-1})\sqrt{n}\Omega\widehat{Q}_n - E^{*-1}\sqrt{n}R(\theta^*) + o_{\mathbb{P}}(\sqrt{n})$. Given that $\sqrt{n}\Omega\widehat{Q}_n \overset{\text{A}}{\sim} \mathcal{N}(0, \Omega\widetilde{B}^*\Omega)$, it follows that

$$\sqrt{n}\ddot{\lambda}_n + E^{*-1}\sqrt{n}R(\theta^*) \overset{\text{A}}{\sim} \mathcal{N}(0, E^{*-1}D^*\widetilde{C}^*D^{*\prime}E^{*-1}), \tag{A.9}$$

so that, if $\mathbb{H}_o$ holds, $R(\theta^*) = 0$ and

$$n\ddot{\lambda}_n'\{E^{*-1}D^*\widetilde{C}^*D^{*\prime}E^{*-1}\}^{-1}\ddot{\lambda}_n \overset{\text{A}}{\sim} \mathcal{X}_r^2. \tag{A.10}$$

Note that $\{E^{*-1}D^*\widetilde{C}^*D^{*\prime}E^{*-1}\}^{-1} = E^*(D^*\widetilde{C}^*D^{*\prime})^{-1}E^* = D^*(\Omega A^*)^{-1}D^{*\prime}(D^*\widetilde{C}^*D^{*\prime})^{-1}D^* (\Omega A^*)^{-1}$

$D^{*\prime}$ using the fact that $E^* := -D^*(\Omega A^*)^{-1}D^{*\prime}$. Therefore,

$$\begin{aligned} n\ddot{\lambda}_n'\{E^{*-1}D^*\widetilde{C}^*D^{*\prime}E^{*-1}\}^{-1}\ddot{\lambda}_n &= n\ddot{\lambda}_n'D^*(\Omega A^*)^{-1}D^{*\prime}(D^*\widetilde{C}^*D^{*\prime})^{-1}D^*(\Omega A^*)^{-1}D^{*\prime}\ddot{\lambda}_n \\ &= n\ddot{\lambda}_n'\ddot{D}_n(\Omega A^*)^{-1}\ddot{D}_n'(\ddot{D}_n\ddot{C}_n\ddot{D}_n')^{-1}\ddot{D}_n(\Omega A^*)^{-1}\ddot{D}_n\ddot{\lambda}_n + o_{\mathbb{P}}(1) \\ &= n\ddot{Q}_n'A^{*-1}\ddot{D}_n'(\ddot{D}_n\ddot{C}_n\ddot{D}_n')^{-1}\ddot{D}_nA^{*-1}\ddot{Q}_n + o_{\mathbb{P}}(1), \end{aligned}$$

where the penultimate equality follows because $\ddot{D}_n \overset{\mathbb{P}}{\to} D^*$ and $\ddot{B}_n \overset{\mathbb{P}}{\to} B^*$ under $\mathbb{H}_o$, as implied by Lemma 5 and the consistency of $\widetilde{A}_n$ for A^*. The last equality follows from (A.7) and the fact that Ω is a diagonal matrix. Note that this final expression is asymptotically equivalent to the definition of $\mathcal{L}\ddot{\mathcal{M}}_n$. So the desired result now follows from (A.10).

Under $\mathbb{H}_a$, note that $\Omega\ddot{Q}_n + \ddot{D}_n'\ddot{\lambda}_n = o_{\mathbb{P}}(1)$ from (A.7) and $\sqrt{n}\ddot{\lambda}_n = O_{\mathbb{P}}(\sqrt{n})$ from (A.9), so that $\sqrt{n}\ddot{Q}_n = O_{\mathbb{P}}(\sqrt{n})$. Furthermore, $\ddot{D}_n = O_{\mathbb{P}}(1)$ and $\ddot{B}_n = O_{\mathbb{P}}(1)$ from Assumption 18, implying that $\mathcal{L}\ddot{\mathcal{M}}_n = O_{\mathbb{P}}(n)$. Therefore, if $c_n = o(n)$, $\lim_{n\to\infty}\mathbb{P}(\mathcal{L}\ddot{\mathcal{M}}_n \geq c_n) = 1$.

(*ii.c*) Given stochastic differentiability of $\widehat{q}_n(\cdot)$ in the sense of Pollard (1985, theorem 5), we can apply a second-order Taylor expansion around $\widetilde{\theta}_n$, so that $2n\{\widehat{q}_n(\ddot{\theta}_n) - \widehat{q}_n(\widetilde{\theta}_n)\} = n(\ddot{\theta}_n - \widetilde{\theta}_n)'\Omega A^*(\ddot{\theta}_n - \widetilde{\theta}_n) + o_{\mathbb{P}}(1)$ using the fact that the stochastic second derivative of $\widehat{q}_n(\cdot)$ is asymptotically equal to ΩA^* at θ^*. The proof of Lemma 5 already showed that $\sqrt{n}(\ddot{\theta}_n - \theta^*) - (\Omega A^*)^{-1}\sqrt{n}\Omega\ddot{Q}_n = \{(\Omega A^*)^{-1}D^*E^{*-1}D^{*\prime}(\Omega A^*)^{-1}\}\sqrt{n}\Omega$ $\ddot{Q}_n + ((\Omega A^*)^{-1}D^*E^{*-1})\sqrt{n}R(\theta^*) + o_{\mathbb{P}}(\sqrt{n})$, from which we further note that $(\Omega A^*)^{-1}\sqrt{n}\Omega\ddot{Q}_n = A^{*-1}$ $\sqrt{n}\ddot{Q}_n = (\widetilde{\theta}_n - \theta^*) + o_{\mathbb{P}}(1)$ as implied by Theorem 6. It follows that

$$\sqrt{n}(\ddot{\theta}_n - \widetilde{\theta}_n) = \{(\Omega A^*)^{-1}D^*E^{*-1}D^{*\prime}(\Omega A^*)^{-1}\}\sqrt{n}\Omega\ddot{Q}_n + ((\Omega A^*)^{-1}D^*E^{*-1})\sqrt{n}R(\theta^*) + o_{\mathbb{P}}(\sqrt{n}).$$

Hence, if $\mathbb{H}_o$ holds,

$$\begin{aligned} 2n\{\widehat{q}_n(\ddot{\theta}_n) - \widehat{q}_n(\widetilde{\theta}_n)\} &= n\ddot{Q}_n'\Omega(\Omega A^*)^{-1}D^{*\prime}E^{*-1}\{D^*(\Omega A^*)^{-1}D^{*\prime}\}E^{*-1}D^*(\Omega A^*)^{-1}\Omega\ddot{Q}_n + o_{\mathbb{P}}(1) \\ &= n\ddot{Q}_n'A^{*-1}D^{*\prime}\{D^*(\Omega A^*)^{-1}D^{*\prime}\}^{-1}D^*A^{*-1}\ddot{Q}_n + o_{\mathbb{P}}(1), \end{aligned}$$

since $E^* := -D^*(\Omega A^*)^{-1}\, D^{*\prime}$. We further note that $\sqrt{n}D^*A^{*-1}\ddot{Q}_n \Rightarrow \widetilde{W} \sim \mathcal{N}(0, D^*A^{*-1}\widetilde{B}^*A^{*-1}D^{*\prime})$. It therefore follows that $\mathcal{Q}\ddot{\mathcal{L}}\mathcal{R}_n := 2n\{\widehat{q}_n(\ddot{\theta}_n) - \widehat{q}_n(\widetilde{\theta}_n)\} \Rightarrow \widetilde{W}'\{D^*(\Omega A^*)^{-1}D^{*\prime}\}^{-1}\widetilde{W}$ under $\mathbb{H}_o$, as desired.

Under $\mathbb{H}_a$, $\sqrt{n}(\ddot{\theta}_n - \widetilde{\theta}_n) = O_{\mathbb{P}}(\sqrt{n})$ since $\{(\Omega A^*)^{-1}D^{*\prime}E^{*-1}D^*(\Omega A^*)^{-1}\}\sqrt{n}\Omega\ddot{Q}_n = O_{\mathbb{P}}(1)$ and $R(\theta^*) \neq 0$, so that $\mathcal{Q}\ddot{\mathcal{L}}\mathcal{R}_n = O_{\mathbb{P}}(n)$. Therefore, if $c_n = o(n)$, $\lim_{n\to\infty}\mathbb{P}(\mathcal{Q}\ddot{\mathcal{L}}\mathcal{R}_n \geq c_n) = 1$. This completes the proof. ■

A.2 Supplementary empirical applications

This section provides additional empirical material for Section 8. First, we provide the estimated $\rho_\tau(\cdot)$ for each group classified by gender and education. Using quadratic, cubic and quartic models for $x_\tau(\cdot)$, Figure A.1 plots the estimated LIPs using work experiences over 0–40 years, and Figure A.2 plots the estimated

LIPs using work experience over 10–40 years. The red, blue, and green lines in the figures denote the fitted LIPs obtained by the quadratic, cubic, and quartic specifications, respectively. The (three colored) curves at the top and the curves at the bottom of each figure are the estimated quantile LIPs for $\tau = 0.75$ and $\tau = 0.25$, respectively. The (three colored) curves in the middle of each figure are the median quantile functions for $\tau = 0.5$. As is apparent in the two figures, the shapes of the estimated quantile curves differ between Figures A.1 and A.2. In particular, the curves in Figure A.2 generally have less curvature and are closer to linearity than those of Figure A.1 which show different patterns depending on the polynomial specification. Further, the fitted quantile functions differ among the polynomial function specification. This feature indicates that the overall shape of the quantile function curve requires a reasonable degree of nonlinearity to accommodate the irregular patterns of the first 10 experience years in the income profiles.

Second, we report the estimation errors measured by $q_{\tau n}(\widehat{\theta}_\tau)$ in each group specification, capturing the value of the criterion function (2) at the estimate $\widehat{\theta}_\tau$. Tables A.1 and A.2 display the errors in the estimated LIPs using work experiences over 0–40 years and 10–40 years, respectively. As shown in the tables, the quartic specification provides the smallest $q_{\tau n}(\widehat{\theta}_\tau)$, and the quadratic specification yields the largest $q_{\tau n}(\widehat{\theta}_\tau)$ among the three specifications. Nonetheless, the quadratic, cubic, and quartic models yield similar estimation errors overall. In the lower panel of each table, we also report $q_{\tau n}(\widehat{\theta}_\tau)$ computed using the rescaled income paths that are obtained by dividing each individual LIP with its integral over the entire working experience profile. As in the nonscaled data case, the estimation errors decline as the degree of the polynomial function rises, although the overall results remain similar.

Estimated errors of the quantiles of the original log income path

		Male			Female		
		Quadratic	Cubic	Quartic	Quadratic	Cubic	Quartic
$\tau = 0.25$	w/o Degree	12.00	11.98	11.96	11.13	11.12	11.10
	Bachelor	10.82	10.77	10.67	10.25	10.20	10.10
	Master	10.93	10.70	10.59	9.91	9.80	9.68
	Ph.D	10.62	10.35	10.20	10.65	10.55	10.43
$\tau = 0.5$	w/o Degree	14.23	14.23	14.20	13.92	13.92	13.89
	Bachelor	13.52	13.39	13.27	12.87	12.80	12.67
	Master	13.59	13.28	13.16	12.22	12.06	11.89
	Ph.D	13.59	13.31	13.14	13.56	13.39	13.26
$\tau = 0.75$	w/o Degree	11.04	11.03	11.01	11.06	11.05	11.01
	Bachelor	10.86	10.66	10.61	10.29	10.21	10.11
	Master	10.85	10.58	10.52	9.78	9.66	9.54
	Ph.D	11.40	11.04	11.04	10.80	10.60	10.55

Estimated errors of the quantiles of the rescaled log income path

		Male			Female		
		Quadratic	Cubic	Quartic	Quadratic	Cubic	Quartic
$\tau = 0.25$	w/o Degree	5.87	5.85	5.76	5.63	5.63	5.56
	Bachelor	6.13	6.10	5.86	6.03	6.02	5.85
	Master	6.47	6.35	6.03	6.21	6.16	5.88
	Ph.D	6.45	6.27	5.85	6.58	6.54	6.22
$\tau = 0.5$	w/o Degree	6.83	6.83	6.77	6.65	6.65	6.59
	Bachelor	7.11	7.07	6.80	6.95	6.95	6.76
	Master	7.56	7.38	6.99	7.15	7.07	6.83
	Ph.D	7.53	7.28	6.79	7.50	7.48	7.21
$\tau = 0.75$	w/o Degree	5.20	5.20	5.19	5.04	5.02	5.01
	Bachelor	5.38	5.29	5.10	5.25	5.19	5.05
	Master	5.72	5.50	5.23	5.32	5.20	5.06
	Ph.D	5.70	5.46	5.14	5.62	5.51	5.36

Table A.1: ESTIMATION ERRORS USING FUNCTION DATA OVER 0 TO 40 WORK EXPERIENCE YEARS. This table shows the estimation errors of the original and rescaled log income paths using the quadratic, cubic and quartic models for each group of the workers classified according to their education levels and genders.

Estimated errors of the quantiles of the log income path

		Male			Female		
		Quadratic	Cubic	Quartic	Quadratic	Cubic	Quartic
$\tau = 0.25$	w/o Degree	8.97	8.94	8.94	8.12	8.10	8.10
	Bachelor	7.92	7.89	7.89	7.31	7.29	7.28
	Master	7.91	7.90	7.90	7.07	7.06	7.06
	Ph.D	7.65	7.63	7.62	7.57	7.56	7.55
$\tau = 0.5$	w/o Degree	10.65	10.64	10.63	10.19	10.18	10.18
	Bachelor	9.89	9.87	9.86	9.27	9.25	9.24
	Master	9.89	9.88	9.87	8.74	8.72	8.72
	Ph.D	9.93	9.90	9.89	9.82	9.80	9.80
$\tau = 0.75$	w/o Degree	8.28	8.27	8.27	8.10	8.09	8.09
	Bachelor	7.89	7.88	7.88	7.47	7.46	7.45
	Master	7.84	7.82	7.82	7.13	7.11	7.11
	Ph.D	8.32	8.32	8.32	7.95	7.95	7.95

Estimated errors of the quantiles of the rescaled log income path

		Male			Female		
		Quadratic	Cubic	Quartic	Quadratic	Cubic	Quartic
$\tau = 0.25$	w/o Degree	3.99	3.90	3.89	3.86	3.79	3.78
	Bachelor	3.87	3.81	3.80	3.90	3.85	3.84
	Master	3.79	3.74	3.74	3.81	3.75	3.75
	Ph.D	3.70	3.65	3.63	3.82	3.76	3.75
$\tau = 0.5$	w/o Degree	4.68	4.63	4.61	4.61	4.57	4.56
	Bachelor	4.54	4.49	4.48	4.57	4.54	4.53
	Master	4.43	4.39	4.38	4.44	4.42	4.41
	Ph.D	4.34	4.30	4.28	4.56	4.53	4.52
$\tau = 0.75$	w/o Degree	3.54	3.53	3.51	3.50	3.49	3.48
	Bachelor	3.43	3.42	3.41	3.46	3.46	3.45
	Master	3.35	3.35	3.34	3.32	3.33	3.32
	Ph.D	3.27	3.26	3.25	3.46	3.46	3.45

Table A.2: ESTIMATION ERRORS USING FUNCTION DATA OVER 10 TO 40 WORK EXPERIENCE YEARS. This table shows the estimation errors of the original and rescaled log income paths under the quadratic, cubic, and quartic for each group of the workers classified according to their education levels and genders.

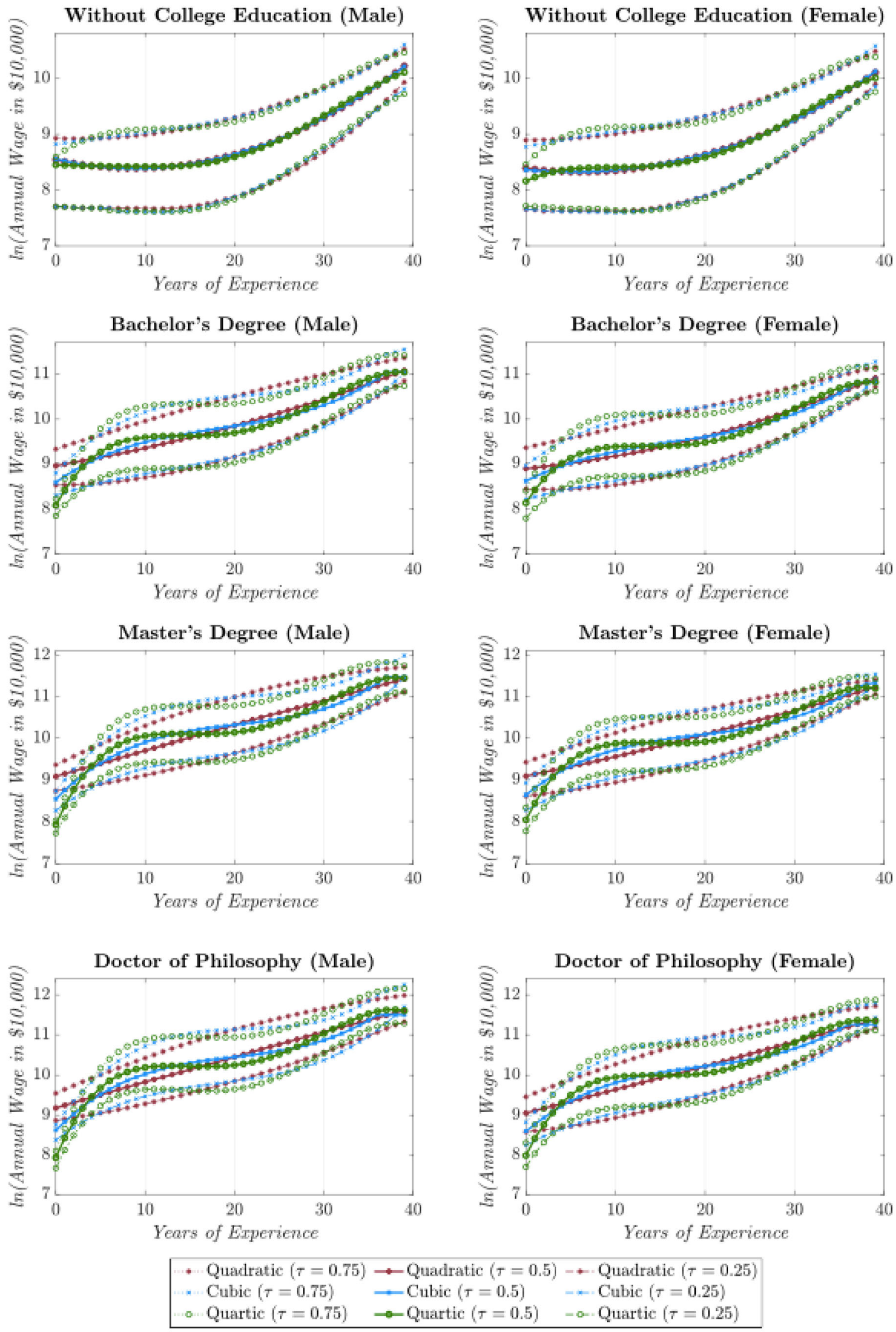

Figure A.1: ESTIMATED QUANTILE FUNCTIONS OVER 0 TO 40 WORK EXPERIENCE YEARS USING THE ORIGINAL LIPS.

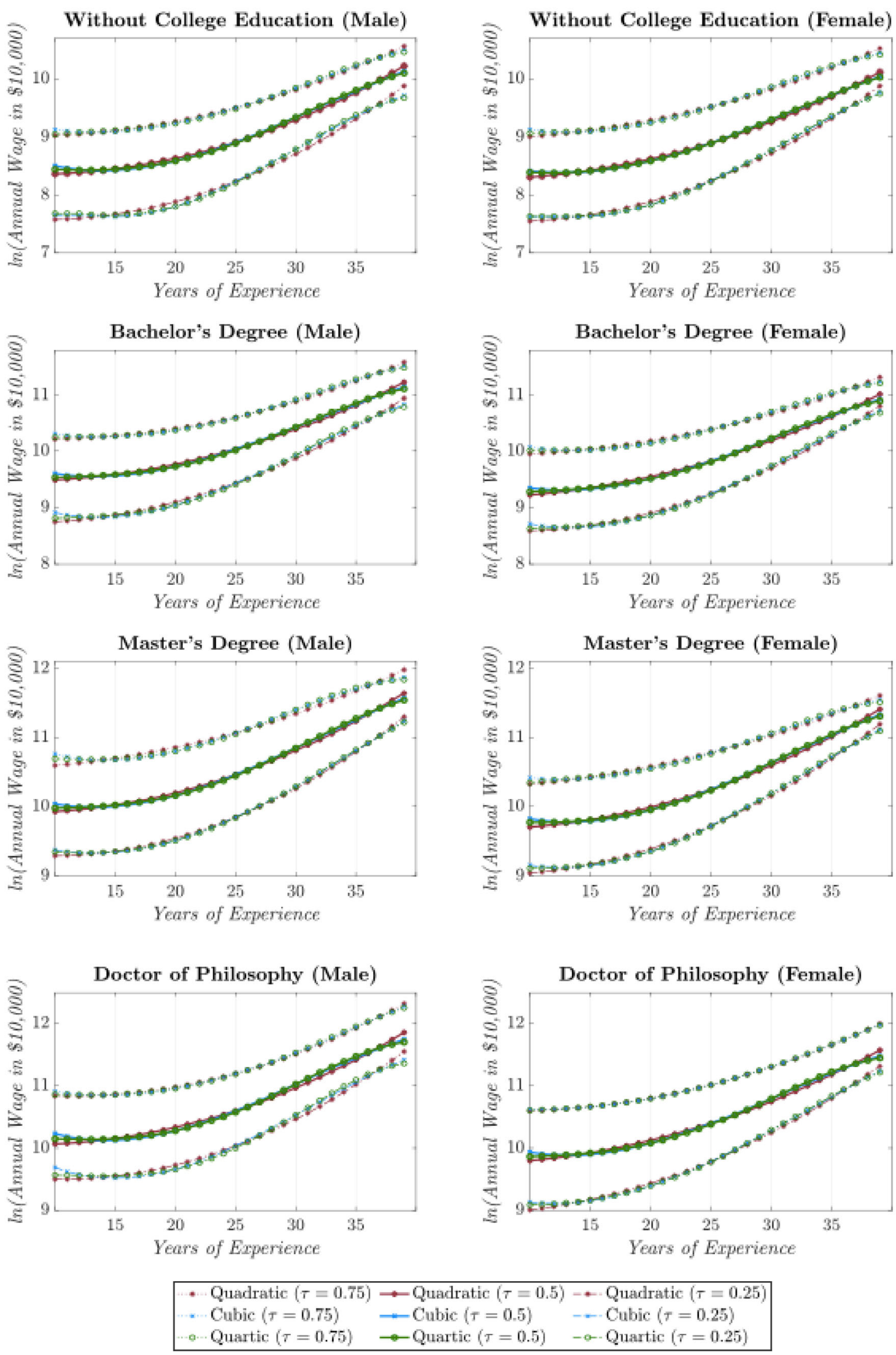

Figure A.2: ESTIMATED QUANTILE FUNCTIONS OVER 10 TO 40 WORK EXPERIENCE YEARS USING THE ORIGINAL LIPS.