Online Supplement to “Pythagorean Generalization of Testing the Equality of Two Symmetric Positive Definite Matrices”

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Abstract

This supplement provides statements and proofs of various subsidiary lemmas and proofs of the main results given in the text of “Pythagorean Generalization of Testing the Equality of Two Symmetric Positive Definite Matrices” by Cho and Phillips (2017).

1 Preliminary Lemmas and Proofs

1.1 Subsidiary Claims and Asymptotic Approximations

Lemma A1. *Given Assumption A,*

(i) $\widehat{\tau}_n = \tau_* + k^{-1}\mathrm{tr}[L_n D_*'] + O_{\mathbb{P}}(n^{-1})$*;*

(ii) $\widehat{\delta}_n = \delta_* + k^{-1}\det[D_*]^{1/k}\mathrm{tr}[L_n] + O_{\mathbb{P}}(n^{-1})$*;*

(iii) $\widehat{\eta}_n = \eta_* + k^{-1}\mathrm{tr}[L_n \widetilde{D}_*']/(k^{-1}\mathrm{tr}[\widetilde{D}_*])^2 + O_{\mathbb{P}}(n^{-1})$*;*

(iv) $\widehat{\sigma}_n = \sigma_* + k^{-1}\mathrm{tr}[L_n D_*'] - k^{-1}\det[D_*]^{1/k}\mathrm{tr}[L_n] + O_{\mathbb{P}}(n^{-1})$*;*

(v) $\widehat{\xi}_n = \xi_* + k^{-1}\mathrm{tr}[L_n D_*'] - k^{-1}\mathrm{tr}[L_n \widetilde{D}_*']/(k^{-1}\mathrm{tr}[\widetilde{D}_*])^2 + O_{\mathbb{P}}(n^{-1})$*; and*

(vi) $\widehat{\gamma}_n = \gamma_* + k^{-1}\det[D_*]^{1/k}\mathrm{tr}[L_n] - k^{-1}\mathrm{tr}[L_n \widetilde{D}_*']/(k^{-1}\mathrm{tr}[\widetilde{D}_*])^2 + O_{\mathbb{P}}(n^{-1})$. □

Lemma A2. *Given Assumption A,*

(i)

$$\widehat{\tau}_n = \tau_* + k^{-1}\{\mathrm{tr}[L_n(I-U_n)D_*'] + [\mathrm{tr}[(J_{j,n} - P_n A_*^{-1}\partial_j A_*)D_*']]'(\widehat{\theta}_n - \theta_*)\} + k^{-1}(\widehat{\theta}_n - \theta_*)'\nabla_\theta^2 \mathrm{tr}[D_*](\widehat{\theta}_n - \theta_*)/2 + o_{\mathbb{P}}(n^{-1}), \quad (1)$$

where $J_{j,n} := G_{j,n} - H_{j,n}$*;*

(ii)

$$\widehat{\delta}_n = \delta_* + k^{-1}\det[D_*]^{\frac{1}{k}}\{\mathrm{tr}[L_n] + \left(k^{-1}-1\right)\mathrm{tr}[L_n]^2/2 + (\mathrm{tr}[P_n]^2 + \mathrm{tr}[U_n^2] - \mathrm{tr}[W_n^2])/2\} + k^{-1}\det[D_*]^{\frac{1}{k}}[\mathrm{tr}[P_n]\mathrm{tr}[R_{j,*}] + \mathrm{tr}[J_{j,n} + U_n A_*^{-1}\partial_j A_* - W_n B_*^{-1}\partial_j B_*]]'(\widehat{\theta}_n - \theta_*) + k^{-1}\det[D_*]^{\frac{1}{k}-1}(\widehat{\theta}_n - \theta_*)'\nabla_\theta^2 \det[D_*](\widehat{\theta}_n - \theta_*)/2 + o_{\mathbb{P}}(n^{-1}); \quad (2)$$

(iii)

$$\widehat{\eta}_n = \eta_* + k^{-1}\mathrm{tr}[L_n\widetilde{D}_*']/(k^{-1}\mathrm{tr}[\widetilde{D}_*])^2 + (k^{-1}\mathrm{tr}[L_n\widetilde{D}_*'])^2/(k^{-1}\mathrm{tr}[\widetilde{D}_*])^3 - \{k^{-1}\mathrm{tr}[L_n W_n\widetilde{D}_*']\}/(k^{-1}\mathrm{tr}[\widetilde{D}_*])^2 - \{k^{-1}[\mathrm{tr}[(-J_{j,n} + P_n B_*^{-1}\partial_j B_*)\widetilde{D}_*']]'(\widehat{\theta}_n - \theta_*)\}/(k^{-1}\mathrm{tr}[\widetilde{D}_*])^2 - k^{-1}\{(\widehat{\theta}_n - \theta_*)'\nabla_\theta^2 \mathrm{tr}[\widetilde{D}_*](\widehat{\theta}_n - \theta_*)\}/\{2(k^{-1}\mathrm{tr}[\widetilde{D}_*])^2\} + o_{\mathbb{P}}(n^{-1});$$

(iv) If $\mathcal{H}_0$ *holds,* $\widehat{\eta}_n = \widehat{\eta}_n^\star + o_{\mathbb{P}}(n^{-1})$*, where*

$$\widehat{\eta}_n^\star := k^{-1}\mathrm{tr}[K_n] + (k^{-1}\mathrm{tr}[K_n])^2 - k^{-1}\mathrm{tr}[K_n W_n] - k^{-1}[\mathrm{tr}[(-J_{j,n} + M_n B_*^{-1}\partial_j B_*)]]'(\widehat{\theta}_n - \theta_*) - k^{-1}(\widehat{\theta}_n - \theta_*)'\nabla_\theta^2 \mathrm{tr}[\widetilde{D}_*](\widehat{\theta}_n - \theta_*)/2,$$

and $M_n := B_*^{-1}(B_n - A_n)$*.* □

Lemma A3. *Given Assumption A, we have:*

(i) if for all $d > 0$, $B_* \neq dA_*$*,*

(i.a) $\widehat{\xi}_n = \xi_* + k^{-1}\mathrm{tr}[L_n D_*'] - k^{-1}\mathrm{tr}[L_n\widetilde{D}_*']/(k^{-1}\mathrm{tr}[\widetilde{D}_*])^2 + O_{\mathbb{P}}(n^{-1})$*;*

(i.b) $\widehat{\gamma}_n = \gamma_* + k^{-1}\det[D_*]^{1/k}\mathrm{tr}[L_n] - k^{-1}\mathrm{tr}[L_n\widetilde{D}_*']/(k^{-1}\mathrm{tr}[\widetilde{D}_*])^2 + O_{\mathbb{P}}(n^{-1})$*;*

(ii) if for some $d_* > 0$, $B_* = d_* A_*$*,*

(ii.a) $\widehat{\xi}_n = d_*\left\{k^{-1}\mathrm{tr}[L_n^2] - k^{-2}\mathrm{tr}[L_n]^2\right\} + o_{\mathbb{P}}(n^{-1})$*;*

(ii.b) $\widehat{\gamma}_n = d_*\left\{k^{-1}\mathrm{tr}[L_n^2] - k^{-2}\mathrm{tr}[L_n]^2\right\}/2 + o_{\mathbb{P}}(n^{-1})$*;*

(iii) If in addition $\mathcal{H}_0$ holds,

(iii.a) $\widehat{\xi}_n = k^{-1}\mathrm{tr}[K_n^2] - k^{-2}\mathrm{tr}[K_n]^2 + o_{\mathbb{P}}(n^{-1})$;

(iii.b) $\widehat{\gamma}_n = \{k^{-1}\mathrm{tr}[K_n^2] - k^{-2}\mathrm{tr}[K_n]^2\}/2 + o_{\mathbb{P}}(n^{-1})$. □

Corollary A1. *Given Assumption A,*

(i) if for all $d > 0$, $B_* \neq dA_*$,

(i.a) $\widehat{\mathfrak{B}}_n^{(1)} = \frac{nk}{2}(\tau_*^2 + 2\sigma_*) + n\{(\tau_* + 1)\mathrm{tr}[D_*{}'L_n] - (\delta_* + 1)\mathrm{tr}[L_n]\} + O_{\mathbb{P}}(1)$;

(i.b) $\widehat{\mathfrak{B}}_n^{(2)} = \frac{nk}{2}(\delta_*^2 + 2\sigma_*) + n\{\mathrm{tr}[D_*{}'L_n] + (\delta_*^2 - 1)\mathrm{tr}[L_n]\} + O_{\mathbb{P}}(1)$;

(i.c) $\widehat{\mathfrak{D}}_n^{(1)} = \frac{nk}{2}(\tau_*^2 + \xi_*) + \frac{n}{2}\{(2\tau_* + 1)\mathrm{tr}[D_*{}'L_n] - (1 + \eta_*)^2\mathrm{tr}[\widetilde{D}_*{}'L_n]\} + O_{\mathbb{P}}(1)$;

(i.d) $\widehat{\mathfrak{D}}_n^{(2)} = \frac{nk}{2}(\eta_*^2 + \xi_*) + \frac{n}{2}\{\mathrm{tr}[D_*{}'L_n] + (2\eta_* - 1)(1 + \eta_*)^2\mathrm{tr}[\widetilde{D}_*{}'L_n]\} + O_{\mathbb{P}}(1)$;

(i.e) $\widehat{\mathfrak{S}}_n^{(1)} = \frac{nk}{2}(\delta_*^2 + 2\gamma_*) + n\{-(1 + \eta_*)^2\mathrm{tr}[\widetilde{D}_*{}'L_n] + (\delta_* + 1)^2\mathrm{tr}[L_n]\} + O_{\mathbb{P}}(1)$;

(i.f) $\widehat{\mathfrak{S}}_n^{(2)} = \frac{nk}{2}(\eta_*^2 + 2\gamma_*) + n\{(\eta_* - 1)(1 + \eta_*)^2\mathrm{tr}[\widetilde{D}_*{}'L_n] + (\delta_* + 1)\mathrm{tr}[L_n]\} + O_{\mathbb{P}}(1)$;

(i.g) $\widehat{\mathfrak{E}}_n^{(1)} = \frac{nk}{2}(\tau_*^2 + 2\gamma_*) + n\{\tau_*\mathrm{tr}[D_*{}'L_n] - (1 + \eta_*)^2\mathrm{tr}[\widetilde{D}_*{}'L_n] + (\delta_* + 1)\mathrm{tr}[L_n]\} + O_{\mathbb{P}}(1)$;

(i.h) $\widehat{\mathfrak{E}}_n^{(2)} = \frac{nk}{2}(\eta_*^2 + 2\sigma_*) + n\{\mathrm{tr}[D_*{}'L_n] + \eta_*(1 + \eta_*)^2\mathrm{tr}[\widetilde{D}_*{}'L_n] - (\delta_* + 1)\mathrm{tr}[L_n]\} + O_{\mathbb{P}}(1)$;

(i.i) $\widehat{\mathfrak{E}}_n^{(3)} = \frac{nk}{2}(\delta_*^2 + \xi_*) + \frac{n}{2}\{\mathrm{tr}[D_*{}'L_n] - (1 + \eta_*)^2\mathrm{tr}[\widetilde{D}_*{}'L_n] + 2\delta_*(\delta_* + 1)\mathrm{tr}[L_n]\} + O_{\mathbb{P}}(1)$. □

Lemma A4. *Given Assumption A, if for some $d_* > 0$, $B_* = d_*A_*$,*

$$\nabla_\theta^2\mathrm{tr}[D_*] + d_*^2\nabla_\theta^2\mathrm{tr}[D_*^{-1}] = 2d_*[\mathrm{tr}[R_{j,*}R_{i,*}]].$$ □

We provide notations relevant to the main claims for the local asymptotic approximations of the tests. We define

$$W_{o,n} := B_*^{-1}(B_n - B_{*,n}); \quad W_{a,n} := B_{*,n}^{-1}(B_n - B_{*,n});$$

$$U_{o,n} := A_*^{-1}(A_n - A_{*,n}); \quad U_{a,n} := A_{*,n}^{-1}(A_n - A_{*,n});$$

$$P_{o,n} := W_{o,n} - U_{o,n}; \quad P_{a,n} := W_{a,n} - U_{a,n};$$

$$M_{o,n} := B_*^{-1}(B_n - A_n - B_{*,n} + A_{*,n}); \quad M_{a,n} := B_{*,n}^{-1}(B_n - A_n - B_{*,n} + A_{*,n});$$

$$L_{o,n} := P_{o,n} + \sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})R_{j,*}; \quad K_{o,n} := M_{o,n} + \sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})S_{j,*};$$

$$L_{a,n} := P_{a,n} + \sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})(B_{*,n}^{-1}\partial_j B_{*,n} - A_{*,n}^{-1}\partial_j A_{*,n});$$

$$K_{a,n} := M_{a,n} + \sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})B_{*,n}^{-1}(\partial_j B_{*,n} - \partial_j A_{*,n}),$$

where $S_{j*} := A_*^{-1}(\partial_j B_* - \partial_j A_*)$. The inverse matrices in $W_{o,n}$, $U_{o,n}$, and $M_{o,n}$ are different from those in $W_{a,n}$, $U_{a,n}$, and $M_{a,n}$, respectively. If the localizing parameters are zero matrices in the inverse matrices, $W_{o,n}$, $U_{o,n}$, and $M_{o,n}$ are reduced versions of $W_{a,n}$, $U_{a,n}$, and $M_{a,n}$. Further note that $A_* = B_*$ under $\mathcal{H}_\ell$, so that $P_{o,n} = M_{o,n}$, $R_{j,*} = S_{j,*}$, and $L_{o,n} = K_{o,n}$. Using this fact, we let

$$\begin{aligned}\widehat{\tau}_{o,n} &:= k^{-1}\mathrm{tr}[K_{o,n}(I - U_{o,n})] \\ &\qquad + k^{-1}[\mathrm{tr}[J_{j,o,n} - M_{o,n}A_*^{-1}\partial_j A_*]]'(\widehat{\theta}_n - \theta_*) + (2k)^{-1}(\widehat{\theta}_n - \theta_*)'\nabla_\theta^2 \mathrm{tr}[D_*](\widehat{\theta}_n - \theta_*);\end{aligned}$$

$$\begin{aligned}\widehat{\delta}_{o,n} &:= k^{-1}\mathrm{tr}[K_{o,n}] + (2k)^{-1}\left(k^{-1} - 1\right)\mathrm{tr}[K_{o,n}]^2 + (2k)^{-1}(\mathrm{tr}[M_{o,n}]^2 + \mathrm{tr}[U_{o,n}^2] - \mathrm{tr}[W_{o,n}^2]) \\ &\qquad + k^{-1}[\mathrm{tr}[J_{j,o,n} + U_{o,n}A_*^{-1}\partial_j A_* - W_{o,n}A_*^{-1}\partial_j B_*]]'(\widehat{\theta}_n - \theta_*) \\ &\qquad\quad + k^{-1}[\mathrm{tr}[M_{o,n}]\mathrm{tr}[S_{j,*}]]'(\widehat{\theta}_n - \theta_*) + (2k)^{-1}(\widehat{\theta}_n - \theta_*)'\nabla_\theta^2 \det[D_*](\widehat{\theta}_n - \theta_*);\end{aligned}$$

$$\begin{aligned}\widehat{\eta}_{o,n} &:= k^{-1}\mathrm{tr}[K_{o,n}] + (k^{-1}\mathrm{tr}[K_{o,n}])^2 - k^{-1}\mathrm{tr}[K_{o,n}W_{o,n}] \\ &\qquad - k^{-1}[\mathrm{tr}[-J_{j,o,n} + M_{o,n}B_*^{-1}\partial_j B_*]]'(\widehat{\theta}_n - \theta_*) - (2k)^{-1}(\widehat{\theta}_n - \theta_*)'\mathrm{tr}[\widetilde{D}_*](\widehat{\theta}_n - \theta_*);\end{aligned}$$

and define $\widehat{\sigma}_{o,n} := \widehat{\tau}_{o,n} - \widehat{\delta}_{o,n}$, $\widehat{\xi}_{o,n} := \widehat{\tau}_{o,n} - \widehat{\eta}_{o,n}$, and $\widehat{\gamma}_{o,n} := \widehat{\delta}_{o,n} - \widehat{\eta}_{o,n}$, where

$$\begin{aligned}J_{j,o,n} &:= G_{j,o,n} - H_{j,o,n} := B_*^{-1}\partial_j(B_n - B_{*,n}) - A_*^{-1}\partial_j(A_n - A_{*,n}); \qquad \text{and} \\ J_{j,a,n} &:= G_{j,a,n} - H_{j,a,n} := B_{*,n}^{-1}\partial_j(B_n - B_{*,n}) - A_{*,n}^{-1}\partial_j(A_n - A_{*,n}).\end{aligned}$$

Lemma A5. *Given Assumption B and $\mathcal{H}_\ell$,*

$$\begin{aligned}(i)\ \ \widehat{\tau}_n - \widehat{\tau}_{o,n} &= n^{-1/2}k^{-1}\mathrm{tr}[V_*] \\ &\quad - n^{-1/2}k^{-1}\mathrm{tr}[F_* W_{o,n} - C_* U_{o,n}] + n^{-1/2}k^{-1}\mathrm{tr}[K_{o,n}V_*] - (nk)^{-1}\mathrm{tr}[C_* V_*] \\ &\qquad + n^{-1/2}k^{-1}[\mathrm{tr}[Q_{j,*} - (F_* B_*^{-1}\partial_j B_* - C_* A_*^{-1}\partial_j A_*)]]'(\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}),\end{aligned}$$

where $Q_{j,*} := B_*^{-1}\partial_j \bar{B}_* - A_*^{-1}\partial_j \bar{A}_*$*;*

(ii) $$\widehat{\delta}_n - \widehat{\delta}_{o,n} = n^{-1/2}k^{-1}\mathrm{tr}[V_*] - n^{-1/2}k^{-1}\mathrm{tr}[F_* W_{o,n} - C_* U_{o,n}] + n^{-1/2}k^{-2}\mathrm{tr}[V_*]\mathrm{tr}[K_{o,n}] + (2nk^2)^{-1}\mathrm{tr}[V_*]^2 + (2nk)^{-1}(\mathrm{tr}[C_*^2] - \mathrm{tr}[F_*^2]) + n^{-1/2}k^{-1}[\mathrm{tr}[Q_{j,*} - (F_* B_*^{-1}\partial_j B_* - C_* A_*^{-1}\partial_j A_*)]]'(\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1});$$

(iii) $$\widehat{\eta}_n - \widehat{\eta}_{o,n} = n^{-1/2}k^{-1}\mathrm{tr}[V_*] - (nk)^{-1}\mathrm{tr}[F_* V_*] - n^{-1/2}k^{-1}\mathrm{tr}[K_{o,n}V_*] + 2(n^{1/2}k^2)^{-1}\mathrm{tr}[V_*]\mathrm{tr}[K_{o,n}] + (nk^2)^{-1}\mathrm{tr}[V_*]^2 - n^{-1/2}k^{-1}\mathrm{tr}[F_* W_{o,n} - \theta_* U_{o,n}] + n^{-1/2}k^{-1}[\mathrm{tr}[Q_{j,*} - (F_* B_*^{-1}\partial_j B_* - C_* A_*^{-1}\partial_j A_*)]]'(\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1});$$

(iv) $$\widehat{\sigma}_n - \widehat{\sigma}_{o,n} = (2k)^{-1}\mathrm{tr}[(K_{o,n} + n^{-1/2}V_*)^2] - (2k^2)^{-1}\mathrm{tr}[K_{o,n} + n^{-1/2}V_*]^2 + (2k^2)^{-1}\mathrm{tr}[K_{a,n}]^2 - (2k)^{-1}\mathrm{tr}[K_{o,n}^2] + o_{\mathbb{P}}(n^{-1});$$

(v) $$\widehat{\xi}_n - \widehat{\xi}_{o,n} = (nk)^{-1}\mathrm{tr}[V_*^2] - 2(n^{1/2}k^2)^{-1}\mathrm{tr}[V_*]\mathrm{tr}[K_{o,n}] + 2(n^{1/2}k)^{-1}\mathrm{tr}[K_{o,n}V_*] - (nk^2)^{-1}\mathrm{tr}[V_*]^2 + o_{\mathbb{P}}(n^{-1});$$

and

(vi) $$\widehat{\gamma}_n - \widehat{\gamma}_{o,n} = (2nk)^{-1}\mathrm{tr}[V_*^2] - (n^{1/2}k^2)^{-1}\mathrm{tr}[V_*]\mathrm{tr}[K_{o,n}] - (2nk^2)^{-1}\mathrm{tr}[V_*]^2 + (n^{1/2}k)^{-1}\mathrm{tr}[K_{o,n}V_*] + o_{\mathbb{P}}(n^{-1}).$$ □

Lemma A6. *Given Assumption B and* $\mathcal{H}_\ell$*,*
(i) $A_{*,n}^{-1} = A_*^{-1} - n^{-1/2}C_* A_*^{-1} + n^{-1}C_*^2 A_*^{-1} + O(n^{-3/2})$*;*
(ii) $B_{*,n}^{-1} = B_*^{-1} - n^{-1/2}F_* B_*^{-1} + n^{-1}F_*^2 B_*^{-1} + O(n^{-3/2})$*;*
(iii) $U_{a,n} = U_{o,n} - n^{-1/2}C_* U_{o,n} + O_{\mathbb{P}}(n^{-3/2})$*;*
(iv) $W_{a,n} = W_{o,n} - n^{-1/2}F_* W_{o,n} + O_{\mathbb{P}}(n^{-3/2})$*;*
(v) $A_{*,n}^{-1}B_{*,n} = I + n^{-1/2}V_* - n^{-1}C_* V_* + O(n^{-3/2})$*;*
(vi) $B_{*,n}^{-1}A_{*,n} = I - n^{-1/2}V_* + n^{-1}F_* V_* + O(n^{-3/2})$*;*
(vii) $P_{a,n} = P_{o,n} - n^{-1/2}(F_* W_{o,n} - C_* U_{o,n}) + O_{\mathbb{P}}(n^{-3/2})$*;*
(viii) $B_{*,n}^{-1}\partial_j B_{*,n} = B_*^{-1}\partial_j B_* + n^{-1/2}(B_*^{-1}\partial_j \bar{B}_* - F_* B_*^{-1}\partial_j B_*) + O(n^{-1})$*;*

(ix) $A_{*,n}^{-1}\partial_j A_{*,n} = A_*^{-1}\partial_j A_* + n^{-1/2}(A_*^{-1}\partial_j \bar{A}_* - C_* A_*^{-1}\partial_j A_*) + O(n^{-1})$*;*

(x) $R_{j,a,*,n} = B_*^{-1}\partial_j B_* - A_*^{-1}\partial_j A_* + n^{-1/2}(Q_{j,*} - (F_* B_*^{-1}\partial_j B_* - C_* A_*^{-1}\partial_j A_*)) + O(n^{-1})$*, where* $R_{j,a,*,n} := B_{n,*}^{-1}\partial_j B_{n,*} - A_{n,*}^{-1}\partial_j A_{n,*}$*;*

(xi) $L_{a,n} = L_{o,n} - n^{-1/2}\{(F_* W_{o,n} - C_* U_{o,n}) - \sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})(Q_{j,*} - (F_* B_*^{-1}\partial_j B_* - C_* A_*^{-1}\partial_j A_*))\} + O_{\mathbb{P}}(n^{-3/2})$*; and*

(xii) $L_{a,n} A_{*,n}^{-1} B_{*,n} = L_{o,n} - n^{-1/2}(F_* W_{o,n} - C_* U_{o,n}) + n^{-1/2}\sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})(Q_{j,*} - (F_* B_*^{-1}\partial_j B_* - C_* A_*^{-1}\partial_j A_*)) + n^{-1/2} L_{o,n}(F_* - C_*) - n^{-1/2}(F_* W_{o,n} - C_* U_{o,n}) + O_{\mathbb{P}}(n^{-3/2})$. □

Lemma A7. *Given Assumption B and* $\mathcal{H}_\ell$*,*

(i) $k^{-1}\mathrm{tr}[B_{*,n}^{-1} A_{*,n}] = 1 - n^{-1/2}k^{-1}\mathrm{tr}[V_*] + n^{-1}k^{-1}\mathrm{tr}[F_* V_*] + O(n^{-3/2})$*;*

(ii) $(k^{-1}\mathrm{tr}[B_{*,n}^{-1} A_{*,n}])^2 = 1 - 2n^{-1/2}k^{-1}\mathrm{tr}[V_*] + 2(nk)^{-1}\mathrm{tr}[F_* V_*] + n^{-1}k^{-2}\mathrm{tr}[V_*]^2 + O(n^{-3/2})$*;*

(iii) $(k^{-1}\mathrm{tr}[B_{*,n}^{-1} A_{*,n}])^{-1} = 1 + n^{-1/2}k^{-1}\mathrm{tr}[V_*] - n^{-1}k^{-1}\mathrm{tr}[F_* V_*] + n^{-1}k^{-2}\mathrm{tr}[V_*]^2 + O(n^{-3/2})$*; and*

(iv) $(k^{-1}\mathrm{tr}[B_{*,n}^{-1} A_{*,n}])^{-2} = 1 + 2n^{-1/2}k^{-1}\mathrm{tr}[V_*] - 2(nk)^{-1}\mathrm{tr}[F_* V_*] + 3n^{-1}k^{-2}\mathrm{tr}[V_*]^2 + O(n^{-3/2})$. □

Lemma A8. *Given Assumption B and* $\mathcal{H}_\ell$ *,*

(i) $\det[A_{*,n}] = \det[A_*]\{1 + n^{-1/2}\mathrm{tr}[C_*] + \frac{1}{2n}(\mathrm{tr}[C_*]^2 - \mathrm{tr}[C_*^2])\} + O(n^{-3/2})$*;*

(ii) $\det[B_{*,n}] = \det[B_*]\{1 + n^{-1/2}\mathrm{tr}[F_*] + \frac{1}{2n}(\mathrm{tr}[F_*]^2 - \mathrm{tr}[F_*^2])\} + O(n^{-3/2})$*;*

(iii) $\det[A_{*,n}]^{-1} = \det[A_*]^{-1}\{1 - n^{-1/2}\mathrm{tr}[C_*] + \frac{1}{2n}(\mathrm{tr}[C_*]^2 + \mathrm{tr}[C_*^2])\} + O(n^{-3/2})$*;*

(iv) $\det[D_{*,n}] = 1 + n^{-1/2}\mathrm{tr}[V_*] + \frac{1}{2n}(\mathrm{tr}[V_*]^2 + \mathrm{tr}[C_*^2] - \mathrm{tr}[F_*^2]) + O(n^{-3/2})$*;*

(v) $\det[D_{*,n}]^{1/k} = 1 + \frac{1}{\sqrt{n}k}\mathrm{tr}[V_*] + \frac{1}{2nk}(\mathrm{tr}[C_*^2] - \mathrm{tr}[F_*^2]) + \frac{1}{2nk^2}\mathrm{tr}[V_*]^2 + O(n^{-3/2})$*; and*

(vi) $\det[D_{*,n}]^{1/k}\mathrm{tr}[L_{a,n}] = \frac{1}{k}\mathrm{tr}[K_{o,n}] + \frac{1}{\sqrt{n}k^2}\mathrm{tr}[V_*]\mathrm{tr}[K_{o,n}] - \frac{1}{\sqrt{n}k}\mathrm{tr}[F_* W_{o,n} - C_* U_{o,n}] + \frac{1}{\sqrt{n}k}[\mathrm{tr}[Q_{j,*} - (F_* B_*^{-1}\partial_j B_* - C_* A_*^{-1}\partial_j A_*)]]'(\widehat{\theta}_n - \theta_*) + O(n^{-3/2})$. □

Lemma A9. *Given Assumption B and* $\mathcal{H}_\ell$*,*

(i) $\widehat{\tau}_{o,n} = k^{-1}\mathrm{tr}[K_{o,n}] + O_{\mathbb{P}}(n^{-1})$*;*

(ii) $\widehat{\delta}_{o,n} = k^{-1}\mathrm{tr}[K_{o,n}] + O_{\mathbb{P}}(n^{-1})$*;*

(iii) $\widehat{\eta}_{o,n} = k^{-1}\mathrm{tr}[K_{o,n}] + O_{\mathbb{P}}(n^{-1})$*;*

(iv) $\widehat{\xi}_{o,n} = k^{-1}\mathrm{tr}[K_{o,n}^2] - (k^{-1}\mathrm{tr}[K_{o,n}])^2 + o_{\mathbb{P}}(n^{-1})$*; and*

(v) $\widehat{\gamma}_{o,n} = (2k)^{-1}\mathrm{tr}[K_{o,n}^2] - (2k^2)^{-1}\mathrm{tr}[K_{o,n}]^2 + o_{\mathbb{P}}(n^{-1})$. □

Lemma A10. *Given Assumption C and* $\mathcal{H}_\ell$*,*

$$\widehat{\lambda}_n - \widehat{\delta}_{o,n} = \frac{1}{\sqrt{n}k}\mathrm{tr}[V_*] - \frac{1}{\sqrt{n}k}\mathrm{tr}[F_* W_{o,n} - C_* U_{o,n}] + \frac{1}{2nk}(\mathrm{tr}[C_*^2] - \mathrm{tr}[F_*^2]) - \frac{1}{2k^2}\mathrm{tr}[K_{o,n}]^2 + \frac{1}{\sqrt{n}k}[\mathrm{tr}[Q_{j,*} - (F_* B_*^{-1}\partial_j B_* - C_* A_*^{-1}\partial_j A_*)]]'(\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}).$$ □

Lemma A11. *If $U_{n,t} \sim$ IID $N(0, A_{n,*})$ or IID $N(0, B_{n,*})$, where $A_{n,*} := B_* + n^{-1/2}\bar{A}_*$, $B_{n,*} := B_* + n^{-1/2}\bar{B}_*$, and B_*, $\bar{A}_*$, and $\bar{B}_*$ are symmetric and positive-definite matrices in $\mathbb{R}^{k\times k}$, (i) $\mathcal{L}_n(B_{n,*}) - \mathcal{L}_n(A_{n,*}) \overset{\mathrm{A}}{\sim} N(-\frac{1}{4}\mathrm{tr}[V_*^2], \frac{1}{2}\mathrm{tr}[V_*^2])$ under $U_{n,t} \sim$ IID $N(0, A_{n,*})$, where $V_* := F_* - C_* := B_*^{-1}(\bar{B}_* - \bar{A}_*)$; (ii) $\mathcal{L}_n(B_{n,*}) - \mathcal{L}_n(A_{n,*}) \overset{\mathrm{A}}{\sim} N(\frac{1}{4}\mathrm{tr}[V_*^2], \frac{1}{2}\mathrm{tr}[V_*^2])$ under $U_{n,t} \sim$ IID $N(0, B_{n,*})$; and IID $N(0, A_{n,*})$ and IID $N(0, B_{n,*})$ are mutually contiguous.* □

1.2 Proofs of the Preliminary Lemmas

Proofs of Lemma A1 follow from lemma 4 of Cho and White (2014; CW, henceforth) and Lemma A2 in our study. The proof is therefore omitted.

Proof of Lemma A2: (*i* and *ii*) Lemma 4 of CW provides the asymptotic expansions.

(*iii*) Lemma 4(*i*) of CW gives the expansion of $\mathrm{tr}[\widehat{B}_n\widehat{A}_n^{-1}]$. We apply this expansion to expand $k^{-1}\mathrm{tr}[\widehat{D}_n^{-1}]$ by simply interchanging the roles of A_n and B_n. That is,

$$\frac{1}{k}\mathrm{tr}[\widehat{D}_n^{-1}] - \frac{1}{k}\mathrm{tr}[D_*^{-1}] = -\frac{1}{k}\mathrm{tr}[L_nB_*^{-1}A_*] + \frac{1}{k}\mathrm{tr}[L_nW_nB_*^{-1}A_*] + \frac{1}{k}[\mathrm{tr}[(-J_{j,n} + P_nB_*^{-1}\partial_jB_*)B_*^{-1}A_*]]'(\widehat{\theta}_n - \theta_*) + \frac{1}{2k}(\widehat{\theta}_n - \theta_*)'\nabla_\theta^2\mathrm{tr}[D_*^{-1}](\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}). \quad (3)$$

We also note that by Taylor expansion of $\frac{1}{x}$ yields that $\frac{1}{x} - \frac{1}{x_0} = -\frac{1}{x_0^2}(x - x_0) + \frac{1}{x_0^3}(x - x_0)^2 + \mathcal{R}$, where $\mathcal{R}$ is the remainder. We now let x and x_0 be $\frac{1}{k}\mathrm{tr}[\widehat{D}_n^{-1}]$ and $\frac{1}{k}\mathrm{tr}[D_*^{-1}]$, respectively and also note that $\widehat{\eta}_n = k/\mathrm{tr}[\widehat{D}_n^{-1}] - 1$ and $\eta_* = k/\mathrm{tr}[D_*^{-1}] - 1$. We finally arrange the terms according to their convergence rates to obtain the desired result.

(*ii*) If $\mathcal{H}_0$ further holds, $\eta_* = 0$, $B_*^{-1}A_* = I$, $L_n = K_n$, $k^{-1}\mathrm{tr}[D_*^{-1}] = 1$, and $P_n = M_n$. If all these equalities are applied to (3), the asymptotic expansion of $\widehat{\eta}_n$ reduces to the desired expansion. ■

Proof of Lemma A3: (*i*) Lemmas A3(*i.a* and *i.b*) immediately follow from Lemma A1(*i*, *ii*, and *iii*).

(*ii*) (*ii.a*) From the fact that $B_* = d_*A_*$, it follows that $\mathrm{tr}[D_*^{-1}] = k/d_*$, $D_* = d_*I$, and $D_*^{-1} = d_*^{-1}I$. We now substitute these into $\widehat{\eta}_n$ in Lemma A2 and obtain

$$\widehat{\eta}_n = d_* - 1 + d_*k^{-1}\mathrm{tr}[L_n] + d_*(k^{-1}\mathrm{tr}[L_n])^2 - d_*k^{-1}[\mathrm{tr}[(-J_{j,n} + P_nB_*^{-1}\partial_jB_*)]]'(\widehat{\theta}_n - \theta_*) - \frac{d_*}{k}\mathrm{tr}[L_nW_n] - \frac{d_*^2}{2k}(\widehat{\theta}_n - \theta_*)'\nabla_\theta^2\mathrm{tr}[D_*^{-1}](\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}). \quad (4)$$

In the same way, we substitute $\mathrm{tr}[D_*^{-1}] = k/d_*$, $D_* = d_* I$, and $D_*^{-1} = d_*^{-1} I$ into (1) and obtain

$$\widehat{\tau}_n = d_* - 1 + d_* k^{-1} \mathrm{tr}[L_n] - d_* k^{-1} \mathrm{tr}[L_n U_n] \\ + \frac{d_*}{k} [\mathrm{tr}[J_{j,n} - P_n A_*^{-1} \partial_j A_*]]' (\widehat{\theta}_n - \theta_*) + \frac{1}{2k} (\widehat{\theta}_n - \theta_*)' \nabla_\theta^2 \mathrm{tr}[D_*] (\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}). \quad (5)$$

Therefore, the asymptotic expansion of $\widehat{\xi}_n$ is obtained as

$$\widehat{\xi}_n := \widehat{\tau}_n - \widehat{\eta}_n = d_* k^{-1} \mathrm{tr}[L_n P_n] + d_* k^{-1} [\mathrm{tr}[P_n R_{j,*}]]' (\widehat{\theta}_n - \theta_*) - d_* k^{-2} \mathrm{tr}[L_n]^2 \\ + \frac{1}{2k} (\widehat{\theta}_n - \theta_*)' \left\{ \nabla_\theta^2 \mathrm{tr}[D_*] + d_*^2 \nabla_\theta^2 \mathrm{tr}[D_*^{-1}] \right\} (\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}). \quad (6)$$

Here, the definition of $P_n := W_n - U_n$ is used to simplify the expression. Given this, note that Lemma A4 implies that $\nabla_\theta^2 \mathrm{tr}[D_*] + d_*^2 \nabla_\theta^2 \mathrm{tr}[D_*^{-1} = 2 d_* \mathrm{tr}[R_{j,*} R_{i,*}]$. Therefore,

$$\widehat{\xi}_n = d_* k^{-1} \mathrm{tr}[L_n P_n] + d_* k^{-1} [\mathrm{tr}[P_n R_{j,*}]]' (\widehat{\theta}_n - \theta_*) - d_* k^{-2} \mathrm{tr}[L_n]^2 \\ + d_* k^{-1} (\widehat{\theta}_n - \theta_*)' [\mathrm{tr}[R_{j,*} R_{i,*}]] (\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}).$$

We recall the definition of $L_n := P_n + \sum_{j=1}^{\ell} (\widehat{\theta}_{j,n} - \theta_{j,*}) R_{j,*}$, and note that this implies

$$\widehat{\xi}_n = d_* k^{-1} \mathrm{tr}[P_n^2] + 2 d_* k^{-1} [\mathrm{tr}[P_n R_{j,*}]]' (\widehat{\theta}_n - \theta_*) - d_* k^{-2} \mathrm{tr}[L_n]^2 \\ + d_* k^{-1} (\widehat{\theta}_n - \theta_*)' [\mathrm{tr}[R_{j,*} R_{i,*}]] (\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}), \quad (7)$$

that is also equal to $d_* k^{-1} \mathrm{tr}[L_n^2] - d_* k^{-2} \mathrm{tr}[L_n]^2 + o_{\mathbb{P}}(n^{-1})$.

(*ii.b*) Note that corollary 5(*ii*) of CW shows that $\widehat{\sigma}_n = -\frac{d_*}{2k^2} \mathrm{tr}[L_n]^2 + \frac{d_*}{2k} \mathrm{tr}[L_n^2] + o_{\mathbb{P}}(n^{-1})$. Also, $\widehat{\gamma}_n := \widehat{\xi}_n - \widehat{\sigma}_n$. Thus, $\widehat{\gamma}_n = 2^{-1} d_* \{k^{-1} \mathrm{tr}[L_n^2] - (k^{-1} \mathrm{tr}[L_n])^2\} + o_{\mathbb{P}}(n^{-1})$ using (*ii.a*).

(*iii*) Given Lemma A3(*ii*), we let $d_* = 1$ and $L_n = K_n$ to complete the proof. ■

Corollary A1 is implied by Lemmas A2, A3, and lemma 5 of CW.

Proof of Lemma A4: By lemma A5(*i*) of CW and the fact that $A_*^{-1} B_* = d_* I$,

$$\partial_{ji}^2 \mathrm{tr}[D_*] = \mathrm{tr}[A_*^{-1} B_* \{(B_*^{-1} \partial_{ji}^2 B_* - A_*^{-1} \partial_{ji}^2 A_*) - (R_{j,*} A_*^{-1} \partial_i A_* + R_{i,*} A_*^{-1} \partial_j A_*)\}] \\ = d_* \mathrm{tr}[(B_*^{-1} \partial_{ji}^2 B_* - A_*^{-1} \partial_{ji}^2 A_*) - (R_{j,*} A_*^{-1} \partial_i A_* + R_{i,*} A_*^{-1} \partial_j A_*)].$$

The asymptotic expansion of $\partial^2_{ji}\mathrm{tr}[D_*^{-1}]$ is also obtained by simply interchanging the roles of A_* and B_*:

$$\begin{aligned}\partial^2_{ji}\mathrm{tr}[D_*^{-1}] &= \mathrm{tr}[B_*^{-1}A_*\{(A_*^{-1}\partial^2_{ji}A_* - B_*^{-1}\partial^2_{ji}B_*) + (R_{j,*}B_*^{-1}\partial_i B_* + R_{i,*}B_*^{-1}\partial_j B_*)\}] \\ &= d_*^{-1}\mathrm{tr}[(A_*^{-1}\partial^2_{ji}A_* - B_*^{-1}\partial^2_{ji}B_*) + (R_{j,*}B_*^{-1}\partial_i B_* + R_{i,*}B_*^{-1}\partial_j B_*)].\end{aligned}$$

Therefore, $\partial^2_{ji}\mathrm{tr}[D_*] + d_*^2\partial^2_{ji}\mathrm{tr}[D_*^{-1}] = 2d_*\mathrm{tr}[R_{j,*}R_{i,*}]$ by noting that $R_{i,*} := B_*^{-1}\partial_i B_* - A_*^{-1}\partial_i A_*$ and $R_{j,*} := B_*^{-1}\partial_j B_* - A_*^{-1}\partial_j A_*$. ■

Proof of Lemma A5: (*i*) We apply lemma 4(*i*) of CW and obtain the following expansion for $\widehat{\tau}_n$:

$$\begin{aligned}\widehat{\tau}_n = \tau_{*,n} &+ \frac{1}{k}\mathrm{tr}[L_{a,n}A_{*,n}^{-1}B_{*,n}] + \frac{1}{k}[\mathrm{tr}[(J_{j,a,n} - P_{a,n}A_*^{-1}\partial_j A_*)A_*^{-1}B_*]]'(\widehat{\theta}_n - \theta_*) \\ &- \frac{1}{k}\mathrm{tr}[L_{a,n}U_{a,n}A_{*,n}^{-1}B_{*,n}] + \frac{1}{2k}(\widehat{\theta}_n - \theta_*)'\nabla^2_\theta\mathrm{tr}[D_*](\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}),\end{aligned}$$

where $\tau_{*,n} := k^{-1}\mathrm{tr}[B_{*,n}A_{*,n}^{-1}] - 1$. We now use Lemma A6(*ii*, *v*, *vii*, and *xii*) for the first three terms and obtain

$$\begin{aligned}\widehat{\tau}_n = &\frac{1}{n^{1/2}k}\mathrm{tr}[V_*] - \frac{1}{nk}\mathrm{tr}[C_*V_*] - \frac{1}{n^{1/2}k}\mathrm{tr}[(F_*W_{o,n} - C_*U_{o,n})] \\ &+ \frac{1}{n^{1/2}k}\mathrm{tr}[L_{o,n}V_*] + \frac{1}{n^{1/2}k}[\mathrm{tr}[(Q_{j,*} - (F_*B_*^{-1}\partial_j B_* - C_*A_*^{-1}\partial_j A_*))]]'(\widehat{\theta}_n - \theta_*) \\ &+ \frac{1}{k}\mathrm{tr}[L_{o,n}] + \frac{1}{k}[\mathrm{tr}[(J_{j,o,n} - P_{o,n}A_*^{-1}\partial_j A_*)A_*^{-1}B_*]]'(\widehat{\theta}_n - \theta_*) \\ &- \frac{1}{k}\mathrm{tr}[L_{o,n}U_{o,n}] + \frac{1}{2k}(\widehat{\theta}_n - \theta_*)'\nabla^2_\theta\mathrm{tr}[D_*](\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}). \quad (8)\end{aligned}$$

Note that $P_{o,n} = M_{o,n}$, $L_{o,n} = K_{o,n}$ under $\mathcal{H}_\ell$ and also that

$$\widehat{\tau}_{o,n} := \frac{1}{k}\mathrm{tr}[K_{o,n}(I - U_{o,n})] + \frac{1}{k}[\mathrm{tr}[J_{j,o,n} - M_{o,n}A_*^{-1}\partial_j A_*]]'(\widehat{\theta}_n - \theta_*) + \frac{1}{2k}(\widehat{\theta}_n - \theta_*)'\nabla^2_\theta\mathrm{tr}[D_*](\widehat{\theta}_n - \theta_*).$$

This represents the last second to the last fourth terms in (8). Substituting $\widehat{\tau}_{o,n}$ into these terms completes the proof.

(*ii*) We apply Lemma 4(*ii*) of CW and obtain the following expansion for $\widehat{\delta}_n$:

$$\widehat{\delta}_n = \delta_{*,n} + \frac{1}{k}\det[D_{*,n}]^{-\frac{1}{k}}\mathrm{tr}[L_{a,n}] + \frac{1}{2k}\det[D_{*,n}]^{-\frac{1}{k}}\left\{\left(\frac{1}{k}-1\right)\mathrm{tr}[L_{a,n}]^2 - \mathrm{tr}[W_{a,n}^2]\right\}$$
$$+ \frac{1}{k}\det[D_{*,n}]^{-\frac{1}{k}}\left\{\frac{1}{2}(\mathrm{tr}[P_{a,n}]^2 + \mathrm{tr}[U_{a,n}^2]) + \mathrm{tr}[P_{a,n}][\mathrm{tr}[R_{j,a,*,n}]]'(\widehat{\theta}_n - \theta_*)\right\}$$
$$+ \frac{1}{k}\det[D_{*,n}]^{-\frac{1}{k}}[\mathrm{tr}[J_{j,o,n} + U_{a,n}A_{*,n}^{-1}\partial_j A_{*,n} - W_{a,n}B_{*,n}^{-1}\partial_j B_{*,n}]]'(\widehat{\theta}_n - \theta_*)$$
$$+ \frac{1}{2k}\det[D_{*,n}]^{\frac{1}{k}-1}(\widehat{\theta}_n - \theta_*)'\nabla_\theta^2\det[D_{*,n}](\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}),$$

where $\delta_{*,n} := \det[B_{*,n}A_{*,n}^{-1}]^{1/k} - 1$. We note that Lemma A7(*v*) implies that $\delta_{*,n} = \frac{1}{\sqrt{n}k}\mathrm{tr}[V_*] + \frac{1}{2nk}(\mathrm{tr}[C_*^2] - \mathrm{tr}[F_*^2]) + \frac{1}{2nk^2}\mathrm{tr}[V_*]^2 + O(n^{-3/2})$, and the asymptotic expansion of $\det[D_{*,n}]^{1/k}\mathrm{tr}[L_{a,n}]$ is given by Lemma A7(*vi*). If we collect all these terms,

$$\widehat{\delta}_n = \frac{1}{\sqrt{n}k}\mathrm{tr}[V_*] + \frac{1}{2nk}(\mathrm{tr}[C_*^2] - \mathrm{tr}[F_*^2]) + \frac{1}{2nk^2}\mathrm{tr}[V_*]^2 + \frac{1}{k}\mathrm{tr}[K_{o,n}] + \frac{1}{\sqrt{n}k^2}\mathrm{tr}[V_*]\mathrm{tr}[K_{o,n}]$$
$$- \frac{1}{\sqrt{n}k}\mathrm{tr}[F_*W_{o,n}] + \frac{1}{\sqrt{n}k}\mathrm{tr}[C_*U_{o,n}] + \frac{1}{2k}(\mathrm{tr}[M_{o,n}]^2 + \mathrm{tr}[U_{o,n}^2]) + \frac{1}{2k}\left(\frac{1}{k}-1\right)\mathrm{tr}[K_{o,n}]^2$$
$$- \frac{1}{2k}\mathrm{tr}[W_{o,n}^2] + \frac{1}{\sqrt{n}k}[\mathrm{tr}[Q_{j,*} - (F_*B_*^{-1}\partial_j B_* - C_*A_*^{-1}\partial_j A_*)]]'(\widehat{\theta}_n - \theta_*)$$
$$+ \frac{1}{k}[\mathrm{tr}[U_{o,n}A_*^{-1}\partial_j A_* - W_{o,n}B_*^{-1}\partial_j B_*]]'(\widehat{\theta}_n - \theta_*) + \frac{1}{k}\mathrm{tr}[M_{o,n}][\mathrm{tr}[S_{j,*}]]'(\widehat{\theta}_n - \theta_*)$$
$$+ \frac{1}{k}[\mathrm{tr}[J_{j,o,n}]]'(\widehat{\theta}_n - \theta_*) + \frac{1}{2k}(\widehat{\theta}_n - \theta_*)' =\nabla_\theta^2\det[D_*](\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}). \quad (9)$$

This equation is derived by using the fact that $L_{o,n} = K_{o,n}$, $R_{j,*} = S_{j,*}$, and $P_{o,n} = M_{o,n}$ under $\mathcal{H}_\ell$. We now note the definition of $\widehat{\delta}_{o,n}$:

$$\widehat{\delta}_{o,n} := \frac{1}{k}\mathrm{tr}[K_{o,n}] + \frac{1}{2k}\left(\frac{1}{k}-1\right)\mathrm{tr}[K_{o,n}]^2 + \frac{1}{2k}(\mathrm{tr}[M_{o,n}]^2 + \mathrm{tr}[U_{o,n}^2] - \mathrm{tr}[W_{o,n}^2])$$
$$+ \frac{1}{k}[\mathrm{tr}[J_{j,o,n} + U_{o,n}A_*^{-1}\partial_j A_* - W_{o,n}A_*^{-1}\partial_j B_*]]'(\widehat{\theta}_n - \theta_*)$$
$$+ \frac{1}{k}[\mathrm{tr}[M_{o,n}]\mathrm{tr}[S_{j,*}]]'(\widehat{\theta}_n - \theta_*) + \frac{1}{2k}(\widehat{\theta}_n - \theta_*)'\nabla_\theta^2\det[D_*](\widehat{\theta}_n - \theta_*).$$

If the right-side terms of (9) that correspond to the definition of $\widehat{\delta}_{o,n}$ are collected into $\widehat{\delta}_{o,n}$, the desired result follows.

(*iii*) Note that Lemma A2(*i*) is simplified into

$$\widehat{\eta}_n = \eta_{*,n} + k^{-1}\mathrm{tr}[L_{a,n}B_{*,n}^{-1}A_{*,n}]/(k^{-1}\mathrm{tr}[D_{*,n}^{-1}])^2 \\ + (k^{-1}\mathrm{tr}[L_{o,n}])^2 - k^{-1}\mathrm{tr}[L_{o,n}W_{o,n}] - k^{-1}[\mathrm{tr}[(-J_{j,n} + P_{o,n}B_*^{-1}\partial_j B_*)B_*^{-1}A_*]]'(\widehat{\theta}_n - \theta_*) \\ - (2k)^{-1}(\widehat{\theta}_n - \theta_*)'\nabla_\theta^2\mathrm{tr}[D_*^{-1}](\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}) \quad (10)$$

under $\mathcal{H}_\ell$, where $\eta_{*,n} := (k^{-1}\mathrm{tr}[B_{*,n}^{-1}A_{*,n}])^{-1} - 1$. Given this, we further note that

$$k^{-1}\mathrm{tr}[L_{a,n}B_{*,n}^{-1}A_{*,n}]/(k^{-1}\mathrm{tr}[D_{*,n}])^2 = k^{-1}\mathrm{tr}[L_{o,n}] - n^{-1/2}k^{-1}\mathrm{tr}[F_*W_{o,n} - C_*U_{o,n}] \\ + n^{-1/2}k^{-1}\sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})[Q_{j,*} - (F_*B_*^{-1}\partial_j B_* - C_*A_*^{-1}\partial_j A_*)] \\ - n^{-1/2}k^{-1}\mathrm{tr}[L_{o,n}V_*] + 2n^{-1/2}k^{-2}\mathrm{tr}[V_*]\mathrm{tr}[L_{o,n}] + o_{\mathbb{P}}(n^{-1}) \quad (11)$$

using Lemmas A4(*vi*, *xi*) and A6(*iv*). If we substitute (11) into (10) and use Lemma A4(*vi*), the the desired result is obtained.

(*iv*) We now use Lemmas A5(*i* and *ii*) and compute $\widehat{\sigma}_n$ by its definition. That is,

$$\widehat{\sigma}_n := \widehat{\tau}_n - \widehat{\delta}_n = \widehat{\sigma}_{o,n} + \frac{1}{2k}\left\{\frac{1}{n}\mathrm{tr}[V_*^2] + \frac{2}{\sqrt{n}}\mathrm{tr}[K_{o,n}V_*] + \mathrm{tr}[K_{o,n}^2]\right\} - \frac{1}{2k}\mathrm{tr}[K_{o,n}^2] \\ - \frac{1}{2k^2}\left\{\frac{1}{n}\mathrm{tr}[V_*]^2 + \frac{2}{\sqrt{n}}\mathrm{tr}[V_*]\mathrm{tr}[K_{o,n}] + \mathrm{tr}[K_{o,n}]^2\right\} + \frac{1}{2k^2}\mathrm{tr}[K_{o,n}]^2 + o_{\mathbb{P}}(n^{-1}).$$

Note further that $\frac{1}{n}\mathrm{tr}[V_*^2] + \frac{2}{\sqrt{n}}\mathrm{tr}[K_{o,n}V_*] + \mathrm{tr}[K_{o,n}^2] = \mathrm{tr}[(K_{o,n} + n^{-1/2}V_*)^2]$ and $\frac{1}{n}\mathrm{tr}[V_*]^2 + \frac{2}{\sqrt{n}}\mathrm{tr}[V_*]$ $\mathrm{tr}[K_{o,n}] + \mathrm{tr}[K_{o,n}]^2 = \mathrm{tr}[K_{o,n} + n^{-1/2}V_*]^2$. Using these facts, we obtain that

$$\widehat{\sigma}_n = \widehat{\sigma}_{o,n} + \frac{1}{2k}\{\mathrm{tr}[(K_{o,n} + n^{-1/2}V_*)^2] - \mathrm{tr}[K_{o,n}^2]\} - \frac{1}{2k^2}\{\mathrm{tr}[K_{o,n} + n^{-1/2}V_*]^2 - \mathrm{tr}[K_{o,n}]^2\} + o_{\mathbb{P}}(n^{-1}).$$

This is the desired result.

(*v*) Note that $\widehat{\xi}_n \equiv \widehat{\tau}_n - \widehat{\eta}_n$ and that the asymptotic approximations of $\widehat{\tau}_n$ and $\widehat{\eta}_n$ are provided in Lemmas A5(*i* and *ii*).

$$\widehat{\xi}_n = \widehat{\tau}_{o,n} - \widehat{\eta}_{o,n} + (nk)^{-1}\mathrm{tr}[C_*^2 - 2C_*F_* + F_*^2] + 2n^{-1/2}k^{-1}\mathrm{tr}[K_{o,n}V_*] \\ - n^{-1}k^{-2}\mathrm{tr}[V_*]^2 - 2n^{-1/2}k^{-2}\mathrm{tr}[V_*]\mathrm{tr}[K_{o,n}] + o_{\mathbb{P}}(n^{-1}). \quad (12)$$

Note that $\mathrm{tr}[C_*^2 - 2C_*F_* + F_*^2] = \mathrm{tr}[(F_* - C_*)^2] = \mathrm{tr}[V_*^2]$. The desired result follows from this.

(*vi*) Note that $\widehat{\gamma}_n \equiv \widehat{\xi}_n - \widehat{\sigma}_n$. Furthermore, the asymptotic approximations of $\widehat{\xi}_n$ and $\widehat{\sigma}_n$ are provided in Lemmas A5(*i* and *iii*). From these, it follows that

$$\widehat{\gamma}_n = \widehat{\xi}_{o,n} - \widehat{\sigma}_{o,n} + (2nk)^{-1}\mathrm{tr}[V_*^2] - n^{-1/2}k^{-1}\mathrm{tr}[V_*]\mathrm{tr}[K_{o,n}] \\ - (2nk^2)^{-1}\mathrm{tr}[V_*]^2 + n^{-1/2}k^{-1}\mathrm{tr}[K_{o,n}V_*] + o_{\mathbb{P}}(n^{-1}).$$

We finally note that $\widehat{\xi}_{o,n} - \widehat{\sigma}_{o,n} = \widehat{\gamma}_{o,n}$ to complete the proof. ■

Proof of Lemma A6: (*i*) Note that $A_{*,n}^{-1} = [I - n^{-1/2}A_*^{-1}(-\bar{A}_*)]^{-1}A_*^{-1}$. For large enough n, $[I - n^{-1/2}A_*^{-1}(-\bar{A}_*)]^{-1} = I - n^{-1/2}A_*^{-1}\bar{A}_* + n^{-1}A_*^{-1}\bar{A}_*A_*^{-1}\bar{A}_* + ...$, which implies that

$$\begin{aligned} A_{*,n}^{-1} &= [I - n^{-1/2}A_*^{-1}(-\bar{A}_*)]^{-1}A_*^{-1} \\ &= A_*^{-1} - n^{-1/2}A_*^{-1}\bar{A}_*A_*^{-1} + n^{-1}A_*^{-1}\bar{A}_*A_*^{-1}\bar{A}_*A_*^{-1} + O(n^{-3/2}) \\ &= A_*^{-1} - n^{-1/2}C_*A_*^{-1} + n^{-1}C_*^2A_*^{-1} + O(n^{-3/2}). \end{aligned}$$

(*ii*) This follows from Lemma A6(*i*) and the symmetric structure between $A_{*,n}$ and $B_{*,n}$.

(*iii*) Note that $U_{a,n} = A_{*,n}^{-1}(A_n - A_{*,n}) = A_*^{-1}(A_n - A_{*,n}) - n^{-1/2}C_*A_*^{-1}(A_n - A_{*,n}) + O_{\mathbb{P}}(n^{-3/2})$ by Lemma A6(*i*). Here, the right side is $U_{o,n} - n^{-1/2}C_*U_{o,n} + O_{\mathbb{P}}(n^{-3/2})$ by the definition of $U_{o,n}$.

(*iv*) This follows from Lemma A6(*iii*) and the symmetric structure between $A_{*,n}$ and $B_{*,n}$.

(*v*) Note that

$$\begin{aligned} A_{*,n}^{-1}B_{*,n} &= (A_*^{-1} - n^{-1/2}C_*A_*^{-1} + n^{-1}C_*^2A_*^{-1} + O(n^{-3/2}))(B_* + n^{-1/2}\bar{B}_*) \\ &= I + n^{-1/2}(A_*^{-1}\bar{B}_* - A_*^{-1}\bar{A}_*) + n^{-1}(C_*^2A_*^{-1}B_* - C_*A_*^{-1}\bar{B}_*) + O(n^{-3/2}) \\ &= I + n^{-1/2}(B_*^{-1}\bar{B}_* - A_*^{-1}\bar{A}_*) - n^{-1}C_*(B_*^{-1}\bar{B}_* - C_*) + O(n^{-3/2}) \end{aligned}$$

by Lemma A6(*i*) and the definition of $B_{*,n}$. We now note that $V_* := B_*^{-1}\bar{B}_* - A_*^{-1}\bar{A}_* = B_*^{-1}\bar{B}_* - C_*$.

(*vi*) This follows from Lemma A6(*v*) and the symmetric structure between $A_{*,n}$ and $B_{*,n}$.

(*vii*) This follows from the definition of $P_{a,n} := W_{a,n} - U_{a,n}$ and Lemmas A6(*iii* and *iv*).

(*viii*) By Lemma A6(*ii*),

$$B_{*,n}^{-1}\partial_j B_{*,n} = (B_*^{-1} - n^{-1/2}F_*B_*^{-1} + n^{-1}F_*^2B_*^{-1} + O(n^{-3/2}))\partial_j(B_* + n^{-1/2}\bar{B}_*)$$
$$= B_*^{-1}\partial_j B_* + n^{-1/2}(B_*^{-1}\partial_j\bar{B}_* - F_*B_*^{-1}\partial_j B_*) + O(n^{-1}).$$

(*ix*) We can apply the proof of Lemma A6(*viii*).

(*x*) Apply Lemmas A6(*viii* and *ix*) to obtain the desired result.

(*xi*) By Lemmas A6(*vii* and *ix*),

$$L_{a,n} = P_{o,n} + \sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})(B_*^{-1}\partial_j B_* - A_*^{-1}\partial_j A_*) - n^{-1/2}(F_*W_{o,n} - C_*U_{o,n})$$
$$+ n^{-1/2}\sum_{j=1}^{\ell}(\widehat{\theta}_{j,n} - \theta_{j,*})(Q_{j,*} - (F_*B_*^{-1}\partial_j B_* - C_*A_*^{-1}\partial_j A_*)) + O_{\mathbb{P}}(n^{-3/2}).$$

The desired result follows from the definition of $L_{o,n}$.

(*xii*) We combine Lemmas A6(*v* and *xi*) and collect the terms according to their convergence rates. This completes the proof. ■

Proof of Lemma A7: (*i*) This immediately follows from Lemma A6(*vi*).

(*ii*) This immediately follows from Lemma A6(*vi*).

(*iii*) Taylor expansion of $1/x$ at $x = 1$ gives $1/x = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \ldots$. We now let x be $k^{-1}\mathrm{tr}[D_{*,n}^{-1}]$ and use Lemma A7(*i*). If the terms are rearranged according to their convergence rates, the desired result follows.

(*iv*) This immediately follows from Lemma A7(*iii*). ■

Proof of Lemma A8: (*i*) By the proof of lemma A2 (*i*) of CW,

$$\det[A_n] - \det[A_*] = \det[A_*]\mathrm{tr}[A_*^{-1}(A_n - A_*)]$$
$$+ \frac{1}{2}\det[A_*]\{\mathrm{tr}[A_*^{-1}(A_n - A_*)]^2 - \mathrm{tr}[A_*^{-1}(A_n - A_*)A_*^{-1}(A_n - A_*)]\} + O_{\mathbb{P}}(n^{-3/2}).$$

We now simply let A_n be $A_{*,n}$ and note that $C_* = A_*^{-1}(A_{*,n} - A_*) = A_*^{-1}\bar{A}_*$ under $\mathcal{H}_\ell$. This yields the desired result.

(*ii*) This immediately follows from Lemma A8(*i*) and the symmetric structure between $A_{*,n}$ and $B_{*,n}$.

(*iii*) Lemma A2(*iii*) of CW shows that $\det[A_n]^{-1} - \det[A_*]^{-1} = -\det[A_*]^{-1}(\mathrm{tr}[U_n] + \frac{1}{2}\mathrm{tr}[U_n]^2 -$

$\frac{1}{2}\mathrm{tr}[U_n^2]) + O_{\mathbb{P}}(n^{-1})$. Under $\mathcal{H}_\ell$, $U_n = C_*$. If we further let their A_n be $A_{*,n}$, then

$$\det[A_{*,n}]^{-1} - \det[A_*]^{-1} = -\det[A_*]^{-1}\{\mathrm{tr}[A_*^{-1}(A_{*,n} - A_*)] + \frac{1}{2}\mathrm{tr}[A_*^{-1}(A_{*,n} - A_*)]^2 - \frac{1}{2}\mathrm{tr}[A_*^{-1}(A_{*,n} - A_*)^2]\} + O_{\mathbb{P}}(n^{-3/2}).$$

The desired result follows by noting that $C_* = A_*^{-1}(A_{*,n} - A_*) = A_*^{-1}\bar{A}_*$.

(*iv*) Note that

$$\det[D_{*,n}] = \det[A_{*,n}]^{-1}\det[B_{*,n}]$$
$$= \left\{1 + \frac{1}{\sqrt{n}}\mathrm{tr}[F_*] + \frac{1}{2n}(\mathrm{tr}[F_*]^2 - \mathrm{tr}[F_*^2])\right\}\left\{1 - \frac{1}{\sqrt{n}}\mathrm{tr}[C_*] + \frac{1}{2n}(\mathrm{tr}[C_*]^2 + \mathrm{tr}[C_*^2])\right\} + O(n^{-3/2}),$$

where the second equality follows from Lemmas A8(*ii* and *iii*) and the fact that $\det[D_*] = 1$ under $\mathcal{H}_\ell$. Thus,

$$\det[D_{*,n}] = 1 + \frac{1}{\sqrt{n}}\mathrm{tr}[F_* - C_*] + \frac{1}{2n}(\mathrm{tr}[F_* - C_*]^2 + \mathrm{tr}[C_*^2] - \mathrm{tr}[F_*^2]) + O(n^{-3/2}).$$

We further note that $V_* := F_* - C_*$ to yield the result.

(*v*) Taylor expansion applied to $\det[D_{*,n}]^{1/k}$ gives

$$\det[D_{*,n}]^{1/k} = \det[D_*]^{1/k} + \frac{1}{k}\det[D_{*,n}]^{1/k-1}\{\det[D_{*,n}] - \det[D_*]\} + \frac{1}{2k}\left(\frac{1}{k} - 1\right)\{\det[D_{*,n}] - \det[D_*]\}^2 + \dots. \quad (13)$$

Lemma A8(*iv*) implies that $\det[D_{*,n}] - \det[D_*] = \frac{1}{\sqrt{n}}\mathrm{tr}[V_*] + \frac{1}{2n}(\mathrm{tr}[V_*]^2 + \mathrm{tr}[C_*^2] - \mathrm{tr}[F_*^2]) + O(n^{-3/2})$ by noting that $\det[D_*] = 1$ under $\mathcal{H}_\ell$. We now substitute this into (13) and arrange the terms according to their convergence rates. This yields the desired result.

(*vi*) To show this, we combine Lemmas A6(*xi*) and A7(*v*) and rearrange the terms according to their convergence rates. This completes the proof. ■

As Lemma A9 is immediately obtained by applying Lemmas A1 and A3(*ii*), we omit the proof.

Proof of Lemma A10: From the definition of $\widehat{\lambda}_n$, if we approximate the log function around unity, it follows that

$$\widehat{\lambda}_n = \widehat{\delta}_n - \frac{1}{2}\widehat{\delta}_n^2 + o_{\mathbb{P}}(n^{-1}).$$

By Lemma A5(*ii*), $\widehat{\delta}_n = n^{-1/2}k^{-1}\mathrm{tr}[V_*] + k^{-1}\mathrm{tr}[K_{o,n}] + O_{\mathbb{P}}(n^{-1})$, so that $\frac{1}{2}\widehat{\delta}_n^2 = (2nk^2)^{-1}\mathrm{tr}[V_*]^2 + n^{-1/2}k^{-2}\mathrm{tr}[K_{o.n}]\mathrm{tr}[V_*] + (2k^2)^{-1}\mathrm{tr}[K_{o,n}]^2 + o_{\mathbb{P}}(n^{-1})$. Therefore, if we combine this result with Lemma A5(*ii*),

$$\widehat{\lambda}_n = \widehat{\delta}_{o,n} + \frac{1}{\sqrt{n}k}\mathrm{tr}[V_*] - \frac{1}{\sqrt{n}k}\mathrm{tr}[F_*W_{o,n} - C_*U_{o,n}] + \frac{1}{2nk}(\mathrm{tr}[C_*^2] - \mathrm{tr}[F_*^2]) - \frac{1}{2k^2}\mathrm{tr}[K_{o,n}]^2 + \frac{1}{\sqrt{n}k}[\mathrm{tr}[Q_{j,*} - (F_*B_*^{-1}\partial_j B_* - C_*A_*^{-1}\partial_j A_*)]]'(\widehat{\theta}_n - \theta_*) + o_{\mathbb{P}}(n^{-1}),$$

as desired. ■

Proof of Lemma A11: (*i*) Note that

$$\mathcal{L}_n(B_{n,*}) - \mathcal{L}_n(A_{n,*}) = -\frac{n}{2}\log(\det[B_{n,*}A_{n,*}^{-1}]) - \frac{n}{2}\mathrm{tr}[(B_{n,*}^{-1} - A_{n,*}^{-1})\widehat{B}_n].$$

Given this, we note that $(\widehat{B}_n - B_*) = O_{\mathbb{P}}(n^{-1/2})$ from that $U_{n,t} \sim$ IID $N(0, A_{n,*})$ and apply Lemmas A6(*i*, *ii*) to obtain that

$$-n\mathrm{tr}[(B_{n,*}^{-1} - A_{n,*}^{-1})\widehat{B}_n] = -n\mathrm{tr}[(B_{n,*}^{-1} - A_{n,*}^{-1})(\widehat{B}_n - B_*)] + n\mathrm{tr}[(B_{n,*}^{-1} - A_{n,*}^{-1})B_*] = \mathrm{tr}[V_*B_*^{-1}\sqrt{n}(\widehat{B}_n - B_*)] - \mathrm{tr}[F_*^2 - C_*^2 - n^{1/2}V_*] + o_{\mathbb{P}}(1).$$

Next, we apply Lemma A8(*iv*) and obtain that

$$-n\log(\det[B_{n,*}A_{n,*}^{-1}]) = -n\log(\det[D_{n,*}]) = -n^{1/2}\mathrm{tr}[V_*] + \frac{1}{2}(\mathrm{tr}[F_*^2] - \mathrm{tr}[C_*^2]) + o(1),$$

implying that

$$\mathcal{L}_n(B_{n,*}) - \mathcal{L}_n(A_{n,*}) = -\frac{1}{4}(\mathrm{tr}[F_*^2] - \mathrm{tr}[C_*^2]) + \frac{1}{2}\mathrm{tr}[V_*B_*^{-1}\sqrt{n}(\widehat{B}_n - B_*)] + o_{\mathbb{P}}(1).$$

Given this, we note that

$$\mathrm{tr}[V_*B_*^{-1}\sqrt{n}(\widehat{B}_n - B_*)] = \mathrm{vec}[B_*^{-1}V_*']'\mathrm{vec}[\sqrt{n}(\widehat{B}_n - B_*)] = \mathrm{vec}[V_*']'(I_k \otimes B_*^{-1})\mathrm{vec}[\sqrt{n}(\widehat{B}_n - B_*)],$$

and we apply the multivariate central limit theorem to $\sqrt{n}(\widehat{B}_n - B_*)$ using the fact that $U_{n,t} \sim$ IID $N(0, A_{n,*})$, so that

$$\mathrm{vec}[\sqrt{n}(\widehat{B}_n - B_*)] \overset{\mathrm{A}}{\sim} N[\mathrm{vec}[\bar{A}_*], (I_{k^2} + P_V)(B_* \otimes B_*)]$$

by the covariance matrix formula of vectorized Wishart radome variable, where P_V is the transposition permutation matrix such that $P_V \text{vec}[\sqrt{n}(\widehat{B}_n - B_*)] = \text{vec}[\sqrt{n}(\widehat{B}_n - B_*)']$. Given that $\sqrt{n}(\widehat{B}_n - B_*)$ is symmetric, $P_V = I_{k^2}$, leading to that $(I_{k^2} + P_V)(B_* \otimes B_*) = 2B_* \otimes B_*$,

$$\begin{aligned}\text{vec}[V_*']'(I_k \otimes B_*^{-1})(I_{k^2} + P_V)(B_* \otimes B_*)(I_k \otimes B_*^{-1})\text{vec}[V_*'] &= \text{vec}[V_*']'(B_* \otimes B_*^{-1})\text{vec}[V_*'] \\ = \text{vec}[V_*']'\text{vec}[B_*^{-1}V_*'B_*] = \text{vec}[V_*']'\text{vec}[B_*^{-1}\bar{B}_*B_*^{-1}B_*] &= \text{vec}[V_*']' = \text{vec}[V_*] = \text{tr}[V_*^2],\end{aligned}$$

and

$$\text{vec}[V_*']'(I_k \otimes B_*^{-1})\text{vec}[\bar{A}_*] = \text{vec}[V_*']'\text{vec}[B_*^{-1}\bar{A}_*] = \text{tr}[V_*C_*].$$

Therefore, it follows that

$$\text{vec}[V_*']'(I_k \otimes B_*^{-1})\text{vec}[\sqrt{n}(\widehat{B}_n - B_*)] \overset{\text{A}}{\sim} N[\text{tr}[V_*C_*], 2\text{tr}[V_*^2]],$$

and

$$\mathcal{L}_n(B_{n,*}) - \mathcal{L}_n(A_{n,*}) = -\frac{1}{4}(\text{tr}[F_*^2] - \text{tr}[C_*^2]) + \frac{1}{2}\text{tr}[V_*B_*^{-1}\sqrt{n}(\widehat{B}_n - B_*)] \overset{\text{A}}{\sim} N\left(-\frac{1}{4}\text{tr}[V_*^2], \frac{1}{2}\text{tr}[V_*^2]\right).$$

(*ii*) Given (*i*), we can apply Le Cam's third lemma.

(*iii*) Given (*i*), the desired result follows from Le Cam's first lemma. ■

2 Proofs of the Main Claims

Proof of Lemma 1: (*i*) If $A = B$, then clearly $\text{tr}[D] = \text{tr}[A^{-1}B] = \text{tr}[I] = k$ and $\text{tr}[D^{-1}] = \text{tr}[B^{-1}A] = \text{tr}[I] = k$. For the converse, note that $k^{-1}\sum_{j=1}^{k}\lambda_j = 1$, where λ_j is the j-th largest eigenvalue of D and so $\text{tr}[D] = k$. In addition, $k^{-1}\text{tr}[D^{-1}] = 1$ implies that $k^{-1}\sum_{j=1}^{k}\lambda_j^{-1} = 1$, so that the harmonic mean of the eigenvalues of D is 1. That is, the arithmetic mean of the eigenvalues is identical to the harmonic mean. Therefore, for some λ, $\lambda = \lambda_1 = \ldots = \lambda_k$. The given condition also implies that $\lambda = 1$. If we now let C be the orthonormal matrix of the eigenvectors of $A^{-1/2}BA^{-1/2}$, $A^{-1/2}BA^{-1/2} = CIC' = I$. Therefore, $A^{-1/2}BA^{-1/2} = I$ implies $A^{1/2}A^{-1/2}BA^{-1/2}A^{1/2} = A^{1/2}A^{1/2}$, which simplifies to $B = A$.

(*ii*) We can combine Lemma 1(*i*) with lemma 1 of CW. ■

Proof of Theorem 1: The proof follows as corollaries of theorem 1 of CW and Lemmas A1 and A3(*iii*). We particularly note that $D_* = \widetilde{D}_* = I$ and $L_n = K_n$ under $\mathcal{H}_0$. From this, all of $\widehat{\mathfrak{B}}_n^{(i)}$, $\widehat{\mathfrak{D}}_n^{(i)}$, $\widehat{\mathfrak{S}}_n^{(i)}$, and $\widehat{\mathfrak{E}}_n^{(j)}$

$(i = 1, 2;\ j = 1, 2, 3)$ are equivalent to $\frac{n}{2}\mathrm{tr}[K_n^2] + o_{\mathbb{P}}(1)$. Furthermore, all components that constitute $\widehat{\mathfrak{M}}_n$ are equivalent to $\frac{n}{2}\mathrm{tr}[K_n^2] + o_{\mathbb{P}}(1)$, so that $\widehat{\mathfrak{M}}_n = \frac{n}{2}\mathrm{tr}[K_n^2] + o_{\mathbb{P}}(1)$ by its construction. ■

Proof of Corollary 1: From the given condition, we obtain that $\frac{n}{2}\mathrm{tr}[K_n^2] \Rightarrow \frac{1}{2}\mathrm{tr}[A_*^{-2}(Z_B - Z_A + (Z_\theta' \otimes I)Q_*)^2] = \frac{1}{2}\mathrm{vec}((Z_B - Z_A + Q_*'(Z_\theta \otimes I))A_*^{-1})'\mathrm{vec}(A_*^{-1}(Z_B - Z_A + (Z_\theta' \otimes I)Q_*))$ from the fact that for general matrices A and B, $\mathrm{tr}[AB] = \mathrm{vec}[A']'\mathrm{vec}[B]$. We also note that

$$\begin{aligned}\mathrm{vec}(A_*^{-1}(Z_B - Z_A + (Z_\theta' \otimes I)Q_*)) &= \mathrm{vec}(A_*^{-1}(Z_B - Z_A)I_k) + \mathrm{vec}(A_*^{-1}(Z_\theta' \otimes I_k)Q_*)) \\ &= (I_k \otimes A_*^{-1})\mathrm{vec}(Z_B - Z_A) + (Q_*' \otimes A_*^{-1})\mathrm{vec}(Z_\theta' \otimes I_k),\end{aligned}$$

and we similarly obtain that

$$\begin{aligned}\mathrm{vec}((Z_B - Z_A + Q_*'(Z_\theta \otimes I))A_*^{-1}) &= \mathrm{vec}(I_k(Z_B - Z_A)A_*^{-1}) + \mathrm{vec}(Q_*'(Z_\theta \otimes I_k)A_*^{-1})) \\ &= (A_*^{-1} \otimes I_k)\mathrm{vec}(Z_B - Z_A) + (A_*^{-1} \otimes Q_*')\mathrm{vec}(Z_\theta \otimes I_k).\end{aligned}$$

From these two equalities, we obtain that

$$\frac{1}{2}\mathrm{vec}((Z_B - Z_A + Q_*'(Z_\theta \otimes I))A_*^{-1})'\mathrm{vec}(A_*^{-1}(Z_B - Z_A + (Z_\theta' \otimes I)Q_*)) = \mathcal{Z}'\Omega_*\mathcal{Z},$$

as desired. ■

Proof of Theorem 2: Before proving the given claims, we note that from Lemmas A5(*i* and *iii*), it follows that

$$\begin{aligned}\widehat{\tau}_n^2 &= (\widehat{\tau}_{o,n} + n^{-1/2}k^{-1}\mathrm{tr}[V_*])^2 + o_{\mathbb{P}}(n^{-1}) = (k^{-1}\mathrm{tr}[K_{o,n}] + n^{-1/2}k^{-1}\mathrm{tr}[V_*] + O_{\mathbb{P}}(n^{-1}))^2 + o_{\mathbb{P}}(n^{-1}) \\ &= (k^{-1}\mathrm{tr}[K_{o,n} + n^{-1/2}V_*]))^2 + o_{\mathbb{P}}(n^{-1}),\end{aligned} \tag{14}$$

and

$$\begin{aligned}\widehat{\eta}_n^2 &= (\widehat{\eta}_{o,n} + n^{-1/2}k^{-1}\mathrm{tr}[V_*])^2 + o_{\mathbb{P}}(n^{-1}) = (k^{-1}\mathrm{tr}[K_{o,n}] + n^{-1/2}k^{-1}\mathrm{tr}[V_*] + O_{\mathbb{P}}(n^{-1}))^2 + o_{\mathbb{P}}(n^{-1}) \\ &= (k^{-1}\mathrm{tr}[K_{o,n} + n^{-1/2}V_*]))^2 + o_{\mathbb{P}}(n^{-1}),\end{aligned} \tag{15}$$

where the second equality holds by Lemmas A9(*i* and *iii*). We also note that from Lemma A5(*ii*),

$$
\begin{aligned}
\widehat{\delta}_n^2 &= (\widehat{\delta}_{o,n} + n^{-1/2}k^{-1}\mathrm{tr}[V_*])^2 + O_{\mathbb{P}}(n^{-3/2}) \\
&= (k^{-1}\mathrm{tr}[K_{o,n}] + n^{-1/2}k^{-1}\mathrm{tr}[V_*] + O_{\mathbb{P}}(n^{-1}))^2 + O_{\mathbb{P}}(n^{-3/2}) \\
&= (k^{-1}\mathrm{tr}[K_{o,n} + n^{-1/2}V_*]))^2 + O_{\mathbb{P}}(n^{-3/2}), \qquad (16)
\end{aligned}
$$

where the second equality holds by Lemma A9(*ii*).

We now first note that $\widehat{\mathfrak{B}}_n^{(1)} := \frac{nk}{2}(\widehat{\tau}_n^2 + 2\widehat{\sigma}_n)$. Therefore,

$$
\begin{aligned}
\widehat{\mathfrak{B}}_n^{(1)} &= \frac{nk}{2}\widehat{\tau}_{o,n}^2 + nk\widehat{\sigma}_{o,n} + \sqrt{n}\mathrm{tr}[V_*]\widehat{\tau}_{o,n} + \frac{1}{2k}\mathrm{tr}[V_*]^2 + \frac{1}{2}\mathrm{tr}[(V_* + \sqrt{n}K_{o,n})^2] \\
&\quad - \frac{1}{2k}\mathrm{tr}[V_* + \sqrt{n}K_{o,n}]^2 + \frac{n}{2k}\mathrm{tr}[K_{o,n}]^2 - \frac{n}{2}\mathrm{tr}[K_{o,n}^2] + o_{\mathbb{P}}(1) = \frac{1}{2}\mathrm{tr}[(V_* + \sqrt{n}K_{o,n})^2] + o_{\mathbb{P}}(1),
\end{aligned}
$$

where the last equality holds by virtue of the definitions of $\widehat{\tau}_{o,n}$ and $\widehat{\sigma}_{o,n}$.

Second, the structure of $\widehat{\mathfrak{B}}_n^{(2)}$ is symmetric to that of $\widehat{\mathfrak{B}}_n^{(1)}$. In the same way, it follows that $\widehat{\mathfrak{B}}_n^{(2)} = \frac{1}{2}\mathrm{tr}[(V_* + \sqrt{n}K_{o,n})^2] + o_{\mathbb{P}}(1)$.

Third, from the definition $\widehat{\mathfrak{D}}_n^{(1)} := \frac{nk}{2}(\widehat{\tau}_n^2 + \widehat{\xi}_n)$, if we combine this with Lemma A9(*iv*) and (14), it follows that

$$
\begin{aligned}
\widehat{\mathfrak{D}}_n^{(1)} &= \frac{k}{2}\{k^{-2}\mathrm{tr}[V_* + \sqrt{n}K_{o,n}]^2 + k^{-1}\mathrm{tr}[(V_* + \sqrt{n}K_{o,n})^2] \\
&\qquad - k^{-2}\mathrm{tr}[V_* + \sqrt{n}K_{o,n}]^2\} + o_{\mathbb{P}}(n^{-1}) = \frac{1}{2}\mathrm{tr}[(V_* + \sqrt{n}K_{o,n})^2] + o_{\mathbb{P}}(n^{-1}).
\end{aligned}
$$

This is the desired result for $\widehat{\mathfrak{D}}_n^{(1)}$.

Fourth, from the fact that (14) has the same asymptotic approximation as that of (15), the asymptotic approximation of $\widehat{\mathfrak{D}}_n^{(2)}$ is identical to that of $\widehat{\mathfrak{D}}_n^{(1)}$.

Fifth, from the definition of $\widehat{\mathfrak{S}}_n^{(1)} := \frac{nk}{2}(\widehat{\delta}_n^2 + 2\widehat{\gamma}_n)$, it follows that

$$
\begin{aligned}
\widehat{\mathfrak{S}}_n^{(1)} &= \frac{k}{2}\{k^{-2}\mathrm{tr}[V_* + \sqrt{n}K_{o,n}]^2 + k^{-1}\mathrm{tr}[(V_* + \sqrt{n}K_{o,n})^2] \\
&\qquad - k^{-2}\mathrm{tr}[V_* + \sqrt{n}K_{o,n}]^2\} + o_{\mathbb{P}}(n^{-1}) = \frac{1}{2}\mathrm{tr}[(V_* + \sqrt{n}K_{o,n})^2] + o_{\mathbb{P}}(n^{-1})
\end{aligned}
$$

by using Lemma A9(*v*) and (15). This is the desired approximation for $\widehat{\mathfrak{S}}_n^{(1)}$.

Sixth, (16) has the same asymptotic approximation as that of (15), and this implies that the asymptotic expansion of $\widehat{\mathfrak{S}}_n^{(2)}$ is identical to that of $\widehat{\mathfrak{S}}_n^{(1)}$.

Seventh, Theorem 2 and (2) in Cho and Phillips (2017) imply that the asymptotic approximations of $\widehat{\mathfrak{E}}_n^{(1)}$, $\widehat{\mathfrak{E}}_n^{(2)}$, and $\widehat{\mathfrak{E}}_n^{(3)}$ are obtained as $\frac{1}{2}\mathrm{tr}[(-V_* - \sqrt{n}K_{o,n})^2] + o_{\mathbb{P}}(1)$ under $\mathcal{H}_\ell$. Therefore, $\widehat{\mathfrak{E}}_n^{(1)}$, $\widehat{\mathfrak{E}}_n^{(2)}$, and $\widehat{\mathfrak{E}}_n^{(3)}$ are asymptotically equivalent to $\widehat{\mathfrak{B}}_n^{(1)}$, $\widehat{\mathfrak{B}}_n^{(2)}$, $\widehat{\mathfrak{D}}_n^{(1)}$, $\widehat{\mathfrak{D}}_n^{(2)}$, $\widehat{\mathfrak{S}}_n^{(1)}$, and $\widehat{\mathfrak{S}}_n^{(2)}$ under $\mathcal{H}_\ell$.

Finally, from these nine facts, all of $\widehat{\mathfrak{B}}_n^{(i)}$, $\widehat{\mathfrak{D}}_n^{(i)}$, $\widehat{\mathfrak{S}}_n^{(i)}$, and $\widehat{\mathfrak{E}}_n^{(j)}$ ($i = 1, 2$; $j = 1, 2, 3$) are equivalent to $\frac{1}{2}\mathrm{tr}[(V_* + \sqrt{n}K_{o,n})^2] + o_{\mathbb{P}}(1)$. Furthermore, all components that constitute $\widehat{\mathfrak{M}}_n$ are equivalent to $\frac{1}{2}\mathrm{tr}[(V_* + \sqrt{n}K_{o,n})^2] + o_{\mathbb{P}}(1)$, so that $\frac{1}{2}\mathrm{tr}[(V_* + \sqrt{n}K_{o,n})^2] + o_{\mathbb{P}}(1)$. ■

Proof of Theorem 3: The claim structures given for the statistics in Corollaries A1(*i.a* – *i.i*) are symmetric and similar. We therefore prove only the claim on $\widehat{\mathfrak{B}}_n^{(1)}$ in (*i.a*), and $\widehat{\mathfrak{D}}_n^{(1)}$ in (*i.c*) to save the space. The others are proved in a similar fashion.

(*i.a*) For $\widehat{\mathfrak{B}}_n^{(1)}$ to have the greatest leading term, it has to be greater than those of the other tests. By (*i*), we compare the leading term of $\widehat{\mathfrak{B}}_n^{(1)}$ with those of other test statistics:

$$\tau_*^2 > \delta_*^2, \tag{17}$$

$$\sigma_* > \gamma_*, \tag{18}$$

$$\tau_*^2 + \sigma_* > \eta_*^2, \tag{19}$$

$$\tau_*^2 + 2\sigma_* > \delta_*^2 + 2\gamma_*, \tag{20}$$

$$\tau_*^2 + 2\sigma_* > \eta_*^2 + 2\gamma_*, \tag{21}$$

$$\sigma_* > \gamma_*, \tag{22}$$

$$\tau_*^2 + \sigma_* > \delta_*^2 + \gamma_* \tag{23}$$

$$\tau_*^2 > \eta_*^2. \tag{24}$$

Each inequality is obtained by letting the leading term of Corollary A1(*i.a*) be greater than the leading terms of Corollaries A1(*i.b*, *i.c*, *i.d*, *i.e*, *i.f*, *i.g*, *i.h*, *i.i*), and the fact that $\xi_* \equiv \sigma_* + \gamma_*$. These 8 inequalities are necessary for the desired condition.

Given this, note that (17), (18), and (24) are the conditions for $\widehat{\mathfrak{B}}_n^{(1)}$ to have the greatest leading term that are given by Theorem 3(*i.a*). This proves sufficiency. For necessity, note that (18) is identical to (22); (17) and (18) imply (20) and (23); (18) and (24) imply (19) and (21). The same proving methodology applies to the proof of (*iii*, *vi*, *vii*, *viii*, and *x*).

(*i.c*) Given the conditions, we note that μ_* is equal to $\frac{nk}{2}(\tau_*^2 + 2\gamma_*)$ that is achieved by $\widehat{\mathfrak{B}}_n^{(1)}$, $\widehat{\mathfrak{D}}_n^{(1)}$, and $\widehat{\mathfrak{E}}_n^{(1)}$. This proves the sufficiency. For necessity, we compare the leading term of $\widehat{\mathfrak{D}}_n^{(1)}$ with those of other test statistics as before:

$$\gamma_* > \sigma_*, \tag{25}$$

$$\tau_*^2 + \gamma_* > \delta_*^2 + \sigma_*, \tag{26}$$

$$\tau_*^2 > \eta_*^2, \tag{27}$$

$$\tau_*^2 + \sigma_* > \delta_*^2 + \gamma_*, \tag{28}$$

$$\tau_*^2 + \sigma_* > \eta_*^2 + \gamma_*, \tag{29}$$

$$\tau_*^2 + \sigma_* > \tau_*^2 + \gamma_*, \tag{30}$$

$$\tau_*^2 > \delta_*^2 \tag{31}$$

$$\tau_*^2 + \gamma_* > \eta_*^2 + \sigma_*. \tag{32}$$

Each inequality is obtained by letting the leading term of Corollary A1(*i.c*) be greater than the leading terms of Corollaries A1(*i.a*, *i.b*, *i.d*, *i.e*, *i.f*, *i.g*, *i.h*, *i.i*). These 8 inequalities are necessary for the desired condition.

Given this, note that (25) and (30) are contradictory, implying that μ_* cannot be uniquely maximized by the leading term of $\widehat{\mathfrak{D}}_n^{(1)}$. We therefore let $\gamma_* = \sigma_*$ and allow for the existence of multiple maximizers. Then, (25)–(32) reduce to the necessary conditions. The same proving methodology applies to other cases.

(*ii*) Given Corollary A1, we note that $D_* = d_* I$ and $\widetilde{D}_* = d_*^{-1} I$. We apply these two equalities to the results in Corollary A1 and obtain the desired results. This completes the proof. ■

Proof of Theorem 4: Using the equality that $\mathfrak{LR}_n = nk(\widehat{\tau}_n - \widehat{\lambda}_n)$ and Lemmas A5(*i*) and A10, we obtain that

$$\begin{aligned}
\mathfrak{LR}_n &= nk(\widehat{\tau}_{o,n} - \widehat{\delta}_{n,o}) + \sqrt{n}\mathrm{tr}[K_{o,n}V_*] - \mathrm{tr}[C_*V_*] - \frac{1}{2}(\mathrm{tr}[C_*^2] - \mathrm{tr}[F_*^2]) + \frac{n}{2k}\mathrm{tr}[K_{o,n}]^2 + o_{\mathbb{P}}(1) \\
&= \frac{n}{2}\mathrm{tr}[K_{o,n}^2] + \sqrt{n}\mathrm{tr}[K_{o,n}V_*] - \mathrm{tr}[C_*V_*] - \frac{1}{2}(\mathrm{tr}[C_*^2] - \mathrm{tr}[F_*^2]) + o_{\mathbb{P}}(1) \\
&= \frac{n}{2}\mathrm{tr}[K_{o,n}^2] + \sqrt{n}\mathrm{tr}[K_{o,n}V_*] + \frac{1}{2}\mathrm{tr}[V_*^2] + o_{\mathbb{P}}(1) \\
&= \frac{1}{2}\mathrm{tr}[(V_* + \sqrt{n}K_{o,n})^2] + o_{\mathbb{P}}(1)
\end{aligned}$$

using the fact that $\widehat{\sigma}_{o,n} := \widehat{\tau}_{o,n} - \widehat{\delta}_{o,n} = -(2k^2)^{-1}\mathrm{tr}[K_{o,n}]^2 + (2k)^{-1}\mathrm{tr}[K_{o,n}^2]$, where the second last holds by the definition of $V_* := F_* - C_*$. Note that the final right side is the desired result. ■

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