Practical Testing for Normal Mixtures

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Abstract

The current study provides the Gaussian versions used to test for normal mixtures. These versions are highly practical as they can directly be used to simulate the asymptotic critical values of standard tests, for example the likelihood-ratio or Lagrange multiplier tests. We investigate testing for two normal mixtures: one having a single variance and two distinct means, and another having a single mean and two different variances. We derive the Gaussian versions for the two models by associating the score functions with the Hermite and generalized Laguerre polynomials, respectively. Additionally, we compare the performance of the likelihood-ratio and Lagrange multiplier tests using the asymptotic critical values.

Key Words: Gaussian version; LR test; LM test; Hermite polynomial; Generalized Laguerre polynomial.

JEL Classifications: C12, C46.

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1 Introduction

Mixture models are popular for empirical analysis, and testing for the mixture hypothesis is crucial for many purposes. For instance, the regime-switching model assumes an autocorrelated mixture model as a model for business cycle (e.g., Hamilton, 1989). As another example, normal mixtures are used as structural models with multiple equilibria resulting from economic behaviors carried out by economic agencies (e.g., Porter, 1983).

Nevertheless, testing for the mixture hypothesis is nonstandard. When testing for the mixture hypothesis naturally, a nuisance parameter is introduced that does not exist under the null of a single distribution (e.g., Davies, 1977, 1987). The null limit distribution of a standard test, such as the likelihood-ratio (LR) and Lagrange multiplier (LM) tests, diverges from a chi-squared distribution due to the presence of the nuisance parameter (e.g., Cho and White, 2007, 2010; Amengual, Bei, Carrasco, and Sentana, 2025). The null limit distributions of tests are characterized by Gaussian processes whose covariance kernels change depending on the model, leading to different critical values. Without access to the critical values, testing the mixture hypothesis becomes impractical due to the need to undertake testing computationally demanding testing procedures such as the bootstrap method.

The primary objective of this study is to provide versions of the Gaussian processes that can be simulated straightforwardly. We investigate two normal mixtures, one with two distinct means and a common variance, and another with a shared mean and two distinct variances. The null limit distributions of the LR and LM tests are described as being characterized by Gaussian processes, with the covariance kernels of these processes differing between the two models. As a result, the two Gaussian versions are useful in obtaining the asymptotic critical values of the tests.

While we concentrate on the two simple normal mixtures, the critical values for large sample size can be used for a wide variety of normal mixture models, provided the kernel structures remain unchanged. As demonstrated by Amengual et al. (2025, proposition 8), the same Gaussian processes are present when testing for a mixture of conditional normals, thereby allowing us to use the same asymptotic critical values.

We achieve the goal by representing the Gaussian process as a series of functions with independent Gaussian random coefficients. When a Gaussian process is represented in this format, it is easy to simulate and can be used to obtain the asymptotic critical values. For the first model, Cho and White (2007) have already provided this version by demonstrating that their Gaussian version has the same covariance kernel as the Gaussian process. In this study, we derive the version analytically from the score of the log-likelihood function and ensure its use for the asymptotic critical values. It is demonstrated that the score function can be expressed as a sequence of orthogonal Hermite polynomials, and the Gaussian version is obtained by applying the large sample theory to each polynomial individually. For the second model, we also derive another version similar to the first model. We derive the Gaussian version of the score analytically from the log-likelihood function and demonstrate that it can be expressed a sequence of orthogonal generalized Laguerre polynomials.

Existing research has provided Gaussian versions for assessing the mixture hypothesis. Cho and White (2010) investigate the LR test for testing the exponential or Weibull mixture hypothesis and derive the null limit distributions as functionals of Gaussian processes, which are distinct from those in this study. They further provide the versions of the Gaussian processes for testing the exponential or Weibull mixture hypothesis. In parallel, Cho and Han (2009) provide Gaussian version to test for the mixture hypothesis of geometric distributions. To our knowledge, it is new to the literature to derive the Gaussian versions analytically using the Hermite or generalized Laguerre polynomials.

This study is structured as follows. In Section 2, we describe the mixture models and derive the versions of the Gaussian processes analytically. In Section 3, we conduct Monte Carlo simulations and affirm the theoretical findings in Section 2. Finally, we provide concluding remarks in Section 4 and contain all mathematical proofs in the Supplement.

2 Gaussian Versions

In this section, we concentrate on testing for two normal mixture models that are frequently employed in empirical studies: one with two distinct means and a single variance, and another with a shared mean and two different variances.

2.1 Normal Mixture with Two Distinct Means and a Single Variance

As the first model, we consider a random variable Y_t that follows the next normal mixture distribution:

$$Y_t \sim \text{ IID} \begin{cases} \mathcal{N}(\mu_{1*}, \sigma_*^2), & \text{w.p. } \pi_*; \\ \mathcal{N}(\mu_{2*}, \sigma_*^2), & \text{w.p. } 1-\pi_*, \end{cases}$$

and the hypothetical data generating process (DGP) is a normal given as $Y_t \sim$ IID $\mathcal{N}(\mu_*, \sigma_*^2)$. This implies that the null hypothesis can be constructed as follows:

$$H_0 : \pi_* = 1 \text{ and } \mu_{1*} = \mu_*; \;\; \pi_* = 0 \text{ and } \mu_{2*} = \mu_*; \;\text{ or }\; \mu_{1*} = \mu_{2*} = \mu_*.$$

The null hypothesis involves an identification problem. If $\pi_* = 0$, then μ_{1*} is not identified. Similarly, if $\pi_* = 1$, then μ_{2*} is not identified. Conversely, if $\mu_{1*} = \mu_{2*} = \mu_*$, then π_* is not identified (e.g., Davies, 1977, 1987).

The mixture hypothesis can be tested by exploiting the standard test principles. Cho and White (2007) consider testing the null hypothesis using the LR test principle:

$$\mathcal{LR}_n^{(1)} := 2\{L_n(\widehat{\pi}_n, \widehat{\mu}_{1n}, \widehat{\mu}_{2n}, \widehat{\sigma}_n^2, \widehat{\sigma}_n^2) - L_n(1, \widehat{\mu}_{0n}, \mu_2, \widehat{\sigma}_{0n}^2, \sigma_2^2)\},$$

where $L_n(\pi, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) := \sum_{t=1}^n \ell_t(\pi, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$,

$$\ell_t(\pi, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) := \log\left(\frac{\pi}{\sqrt{2\pi\sigma_1^2}}\exp\left[-\frac{(Y_t-\mu_1)^2}{2\sigma_1^2}\right] + \frac{1-\pi}{\sqrt{2\pi\sigma_2^2}}\exp\left[-\frac{(Y_t-\mu_2)^2}{2\sigma_2^2}\right]\right),$$

$$(\widehat{\pi}_n, \widehat{\mu}_{1n}, \widehat{\mu}_{2n}, \widehat{\sigma}_n^2) := \underset{\pi,\mu_1,\mu_2,\sigma^2}{\arg\max}\, L_n(\pi, \mu_1, \mu_2, \sigma^2, \sigma^2), \quad \text{and}$$

$$(\widehat{\mu}_{0n}, \widehat{\sigma}_{0n}^2) := \underset{\mu,\sigma^2}{\arg\max}\, L_n(1, \mu, \mu_2, \sigma^2, \sigma_2^2).$$

Here, (μ_2, σ_2^2) is a placeholder.

Due to the identification problem, Cho and White (2007) derive the null limit distribution of the LR test as a functional of a Gaussian process. Denoting the Gaussian process as $\mathcal{G}(\cdot)$, it has the following kernel structure: for each δ_1 and δ_2,

$$\mathbb{E}[\mathcal{G}(\delta_1)] = 0 \quad \text{and} \quad \mathbb{E}[\mathcal{G}(\delta_1)\mathcal{G}(\delta_2)] = \frac{V(\delta_1,\delta_2)}{\sqrt{V(\delta_1,\delta_1)}\sqrt{V(\delta_2,\delta_2)}},$$

where $\delta := \mu - \mu_*$ and

$$V(\delta_1, \delta_2) = \exp(\delta_1\delta_2) - 1 - \delta_1\delta_2 - \frac{1}{2}\delta_1^2\delta_2^2.$$

Using $\mathcal{G}(\cdot)$, they further obtain the null limit distribution of the LR test given as follows:

$$\mathcal{LR}_n^{(1)} \Rightarrow \max\left[\max{}^2[0, G_*], \sup_{\delta\in\Delta}\min{}^2[0, \mathcal{G}(\delta)]\right],$$

where $G_* \sim \mathcal{N}(0,1)$ such that $\mathbb{E}[G_*\mathcal{G}(\delta)] = \delta^4/\sqrt{24V(\delta,\delta)}$ and Δ is the space of δ. Chen and Chen (2001) also demonstrate that the LR test weakly converges to $\sup_{\delta\in\Delta}\min^2[0, \mathcal{G}(\delta)]$ under the null, if σ_*^2 is known.

The null weak limit has a straightforward interpretation. First, $\max^2[0, G_*]$ is the null weak limit of the LR test obtained while testing $\mu_{1*} = \mu_{2*}$. Meanwhile, $\sup_{\delta\in\Delta}\min^2[0, \mathcal{G}(\delta)]$ is the null weak limit obtained while testing the hypotheses: $\pi_* = 1$ and $\mu_{1*} = \mu_*$; or $\pi_* = 0$ and $\mu_{2*} = \mu_*$. By the symmetry of the normal mixture, the last two null weak limits are identical. The null weak limit of the LR test is obtained as the maximum of the three null weak limits by the LR test principle.

Amengual et al. (2025) exploit the LM test principle to test the same hypothesis. They define the following LM test:

$$\mathcal{LM}_n^{(1)} := \max\left[\frac{H_{3,n}^2}{6n} + \min^2\left[0, \frac{H_{4,n}}{\sqrt{24n}}\right], \sup_{\delta\in\Delta}\min^2\left[0, \frac{G_n(\delta)}{\sqrt{V(\delta,\delta)}}\right]\right],$$

where

$$H_{3,n} := \sum_{t=1}^{n} \widehat{Y}_t(\widehat{Y}_t^2 - 3), \quad H_{4,n} := \sum_{t=1}^{n}(3 - 6\widehat{Y}_t^2 + \widehat{Y}_t^4), \quad \widehat{Y}_t := \frac{Y_t - \widehat{\mu}_{0n}}{\sqrt{\widehat{\sigma}_{0n}^2}}, \quad \text{and}$$

$$G_n(\delta) := \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left\{1 - \delta\widehat{Y}_t + \frac{\delta^2}{2}(\widehat{Y}_t^2 - 1) - \exp\left[\frac{-\delta^2}{2} - \delta\widehat{Y}_t\right]\right\}.$$

Here, $G_n(\cdot)$ is the score function obtained by imposing the hypothesis that $\pi_* = 0$, so that the score function is defined as a function of the unidentified parameter, viz., $\delta := \mu - \mu_*$. In addition, $H_{3,n}$ and $H_{4,n}$ are introduced to test the skewness and kurtosis property of a normal distribution.

The null limit distribution of the LM test is also represented using the same Gaussian process. Specifically, Amengual et al. (2025) derive the following null weak limit:

$$\mathcal{LM}_n^{(1)} \Rightarrow \max\left[\text{plim}_{\delta\to 0}\mathcal{G}^2(\delta) + \min^2[0, G_0], \ \sup_{\delta\in\Delta}\min^2[0, \mathcal{G}(\delta)]\right]$$

such that $\mathbb{E}[G_0\mathcal{G}(\delta)] = -\delta^4/\sqrt{24V(\delta,\delta)}$.

The null weak limit of the LM test has a structure parallel to the LR test. First, $\text{plim}_{\delta\to 0}\mathcal{G}^2(\delta) + \min^2[0, G_0]$ is the null weak limit of the LM test testing $\mu_{1*} = \mu_{2*}$. Second, $\sup_{\delta\in\Delta}\min^2[0, \mathcal{G}(\delta)]$ is the null weak limit of the LM test testing $\pi_* = 1$ and $\mu_{1*} = \mu_*$; or $\pi_* = 0$ and $\mu_{2*} = \mu_*$. Amengual et al. (2025) define the LM test in a manner to choose the maximum out of the three LM tests, producing the null weak limits given as above.

All these tests imply that the Gaussian process $\mathcal{G}(\cdot)$ plays a central role when testing for the mixture hypothesis. Nonetheless, it is not straightforward to find the analytical distribution of $\mathcal{G}(\cdot)$. Due to this difficulty, Cho and White (2007) provide a version of $\mathcal{G}(\cdot)$ whose covariance kernel is identical to that of $\mathcal{G}(\cdot)$. That is, if we let

$$\widetilde{\mathcal{G}}(\delta) := \frac{1}{\sqrt{V(\delta,\delta)}}\sum_{j=3}^{\infty}\frac{\delta^j}{\sqrt{j!}}Z_j,$$

where $Z_j \sim \text{IID}\ \mathcal{N}(0,1)$, for any δ_1 and $\delta_2 \in \Delta$, it follows that $\mathbb{E}[\widetilde{\mathcal{G}}(\delta_1)\widetilde{\mathcal{G}}(\delta_2)] = \mathbb{E}[\mathcal{G}(\delta_1)\mathcal{G}(\delta_2)]$. This aspect implies that $Z_3 = \text{plim}_{\delta\to 0}\widetilde{\mathcal{G}}(\delta)$. Therefore, the null limit distribution of the LR test can be obtained by simulating

$$\max\left[\max^2[0, Z_4],\ \sup_{\delta\in\Delta}\min^2[0, \widetilde{\mathcal{G}}(\delta)]\right]. \tag{1}$$

Bostwick and Steigerwald (2014) provide a STATA code for the simulation. For the LM test, we further note that and $-\mathbb{E}[Z_4\widetilde{\mathcal{G}}(\delta)] = -\delta^4/\sqrt{24V(\delta,\delta)}$, which is identical to $\mathbb{E}[G_0\mathcal{G}(\delta)]$. Therefore, its null limit distribution can be obtained by simulating

$$\max\left[Z_3^2 + \min^2[0, -Z_4],\ \sup_{\delta\in\Delta}\min^2[0, \widetilde{\mathcal{G}}(\delta)]\right]. \tag{2}$$

Although Cho and White (2007) obtained the version of $\mathcal{G}(\cdot)$ by simply comparing the covariance kernels of $\mathcal{G}(\cdot)$ and $\widetilde{\mathcal{G}}(\cdot)$, it is possible to obtain the version analytically. In the next theorem, we provide the analytical derivation of $\widetilde{\mathcal{G}}(\cdot)$.

Theorem 1. *Given the assumptions made so far, if the null hypothesis holds,*

$$G_n(\cdot) := \sum_{j=3}^{\infty}\frac{(\cdot)^j}{\sqrt{j!}}\left[-\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{1}{\sqrt{j!}}\left(-\frac{1}{\sqrt{2}}\right)^j H_j\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right)\right] \Rightarrow \widetilde{\mathcal{Z}}(\cdot) := \sum_{j=3}^{\infty}\frac{(\cdot)^j}{\sqrt{j!}}Z_j,$$

where $H_j(\cdot)$ is the j-th degree Hermite polynomial (e.g., Spiegel, 1968, p. 151).

Remarks. (a) Theorem 1 implies that we can define $\widetilde{\mathcal{G}}(\cdot) := \widetilde{\mathcal{Z}}(\cdot)/\sqrt{V(\cdot,\cdot)}$. It is straightforward to obtain that

$$\mathbb{E}[\widetilde{\mathcal{Z}}(\delta_1)\widetilde{\mathcal{Z}}(\delta_2)] = \sum_{j=3}^{\infty}\frac{1}{j!}(\delta_1\delta_2)^j = \exp(\delta_1\delta_2) - 1 - \frac{1}{2}\delta_1\delta_2 = V(\delta_1,\delta_2).$$

Therefore, $\widetilde{\mathcal{G}}(\cdot)$ has the same covariance kernel as $\mathcal{G}(\cdot)$.

(b) The standard normal random variables $Z_3, Z_4, \ldots$ are obtained by applying the central limit theorem (CLT) to the sum of independent Gaussian random variables. That is, for each $j < \infty$,

$$-\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{1}{\sqrt{j!}}\left(-\frac{1}{\sqrt{2}}\right)^j H_j\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right) \Rightarrow Z_j$$

by noting that $\int_{-\infty}^{\infty}H_j^2(x)\exp(-x^2)dx = 2^j j!\sqrt{\pi}$. The independence between Z_j and $Z_{j'}$ ($j \neq j'$) is due to the orthogonality of the Hermite polynomials, i.e., $\int_{-\infty}^{\infty}H_j(x)H_{j'}(x)\exp(-x^2)dx = 0$.

(c) The third and fourth-degree Hermite polynomials are related to the skewness and kurtosis components of the LM test, respectively. Specifically,

$$\frac{H_{3,n}}{\sqrt{6n}} = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{1}{\sqrt{3!}}\left(-\frac{1}{\sqrt{2}}\right)^{3} H_3\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right) \Rightarrow -Z_3 \quad \text{and}$$

$$\frac{H_{4,n}}{\sqrt{24n}} = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{1}{\sqrt{4!}}\left(-\frac{1}{\sqrt{2}}\right)^{4} H_4\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right) \Rightarrow -Z_4.$$

This implies that the test bases formed by the skewness and kurtosis components constitute the test basis that converges to the Gaussian process under the null hypothesis. From this, the LR test differs from the LM test in treating Z_3. The LR test is devised to treat Z_3 as a part of $\widetilde{\mathcal{G}}(\cdot)$ under the null using the fact that $Z_3 = \text{plim}_{\delta\to 0}\widetilde{\mathcal{G}}(\delta)$. Meanwhile, the LM test is devised to accommodate the role of Z_3 explicitly as a part of its null weak limit.

(d) We assumed the a simple mixture of normals, but it can also be used to test for a mixture of conditional normals, as observed by Amengual et al. (2025, proposition 8), where the same Gaussian process is found when testing for a mixture of conditional normals driven by two distinct location parameters.

2.2 Normal Mixture with Two Different Variances and a Single Mean

As the second model, we suppose that Y_t follows the next normal mixture:

$$Y_t \sim \text{ IID}\begin{cases}\mathcal{N}(\mu_*, \sigma_{1*}^2), & \text{w.p. } \pi_*;\\ \mathcal{N}(\mu_*, \sigma_{2*}^2), & \text{w.p. } 1-\pi_*,\end{cases}$$

but the hypothetical DGP condition is given as $Y_t \sim \text{IID}\,\mathcal{N}(\mu_*, \sigma_*^2)$.

This implies that the null hypothesis can be constructed as follows:

$$H_0 : \pi_* = 1 \text{ and } \sigma_{1*}^2 = \sigma_*^2;\ \ \pi_* = 0 \text{ and } \sigma_{2*}^2 = \sigma_*^2;\ \text{ or } \sigma_{1*}^2 = \sigma_{2*}^2 = \sigma_*^2.$$

As before, the joint hypothesis involves an identification problem. If $\pi_* = 1$, then σ_{2*}^2 is not identified. Similarly, if $\pi_* = 0$, then σ_{1*}^2 is not identified. Conversely, if $\sigma_{1*}^2 = \sigma_{2*}^2 = \sigma_*^2$, then π_* is not identified.

As before, we can apply the LR test principle to test the hypothesis. If we let the LR test be

$$\mathcal{LR}_n^{(2)} := 2\{L_n(\widetilde{\pi}_n, \widetilde{\mu}_n, \widetilde{\mu}_n, \widetilde{\sigma}_{1n}^2, \widetilde{\sigma}_{2n}^2) - L_n(1, \widehat{\mu}_{0n}, \widehat{\sigma}_{0n}^2, \sigma_2^2)\},$$

where

$$(\widetilde{\pi}_n, \widetilde{\mu}_n, \widetilde{\sigma}^2_{1n}, \widetilde{\sigma}^2_{2n}) := \arg\max_{\pi,\mu,\sigma_1^2,\sigma_2^2} L_n(\pi, \mu, \mu, \sigma_1^2, \sigma_2^2),$$

Cho and White (2007) obtain its null limit distribution as follows:

$$\mathcal{LR}_n^{(2)} \Rightarrow \sup_{\gamma\in\Gamma} \min{}^2[0, \mathcal{U}(\gamma)],$$

where $\gamma := 1 - \sigma^2/\sigma_*^2$, Γ is the space of γ, and $\mathcal{U}(\cdot)$ is a Gaussian process with the following covariance kernel: for each γ_1 and γ_2 in Γ,

$$\mathbb{E}[\mathcal{U}(\gamma_1)] = 0, \quad \mathbb{E}[\mathcal{U}(\gamma_1)\mathcal{U}(\gamma_2)] = \frac{W(\gamma_1, \gamma_2)}{\sqrt{W(\gamma_1, \gamma_1)}\sqrt{W(\gamma_2, \gamma_2)}},$$

and

$$W(\gamma_1, \gamma_2) = \frac{1}{\sqrt{1-\gamma_1\gamma_2}} - 1 - \frac{1}{2}\gamma_1\gamma_2.$$

Due to the identification problem, the null limit distribution of the LR test is represented as a functional of the Gaussian process. Here, the Gaussian process $\mathcal{U}(\cdot)$ is obtained while testing $\pi_* = 1$ and $\sigma_{1*}^2 = \sigma_*^2$; or $\pi_* = 0$ and $\sigma_{2*}^2 = \sigma_*^2$. Due to the symmetry of the normal mixture, the weak null limits of the LR test are identical under both hypotheses: $\pi_* = 1$ and $\sigma_{1*}^2 = \sigma_*^2$; and $\pi_* = 0$ and $\sigma_{2*}^2 = \sigma_*^2$. The null weak limit of the LR test under the hypothesis that $\sigma_{1*}^2 = \sigma_{2*}^2$ is dominated by $\sup_{\gamma\in\Gamma} \min^2[0, \mathcal{U}(\gamma)]$ with probability 1, resulting in the same maximum as previously found.

By applying the LM test principle, Amengual et al. (2025) define another LM test as follows:

$$\mathcal{LM}_n^{(2)} := \sup_{\gamma\in\Gamma} \min{}^2\left[0, \frac{D_n(\gamma)}{\sqrt{W(\gamma,\gamma)}}\right],$$

where

$$D_n(\gamma) := \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left\{1 + \frac{\gamma}{2}(1-\widehat{Y}_t^2) - \frac{1}{\sqrt{1-\gamma}}\exp\left[-\left(\frac{\gamma}{1-\gamma}\right)\frac{\widehat{Y}_t^2}{2}\right]\right\}.$$

Here, $D_n(\cdot)$ is the score function obtained while testing $\pi_* = 0$ and $\sigma_{2*}^2 = \sigma_*^2$. As σ_{1*}^2 is not identified, the score function is defined as a function of $\gamma := 1 - \sigma^2/\sigma_*^2$. By showing that $D_n(\cdot) \Rightarrow \mathcal{V}(\cdot)$, a zero-mean Gaussian process with the covariance kernel $W(\gamma_1, \gamma_2)$, Amengual et al. (2025) show that

$$\mathcal{LM}_n^{(2)} \Rightarrow \sup_{\gamma\in\Gamma} \min{}^2\left[0, \mathcal{U}(\gamma)\right]$$

under the null hypothesis. This fact implies that the null weak limits of $\mathcal{LR}_n^{(2)}$ and $\mathcal{LM}_n^{(2)}$ are characterized by the same Gaussian process $\mathcal{U}(\cdot)$ and that they are equivalent.

We obtain a version of $\mathcal{U}(\cdot)$ analytically in parallel to Theorem 1. The next theorem provides it.

Theorem 2. *Given the assumptions made so far, if the null hypothesis holds,*

$$D_n(\cdot) = \sum_{j=2}^{\infty}(\cdot)^j \left[-\frac{1}{\sqrt{n}} \sum_{t=1}^{n} L_j^{(-1/2)} \left(\frac{\widehat{Y}_t^2}{2} \right) \right] \Rightarrow \widetilde{\mathcal{V}}(\cdot) := \sum_{j=2}^{\infty}(\cdot)^j \sqrt{\frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(j+1)}} Z_j,$$

where $L_j^{(\alpha)}(\cdot)$ is the j-th degree generalized Laguerre polynomial (e.g., Hochstrasser, 1964, p.775, 22.3.9).

Remarks. (a) If we let $\widetilde{\mathcal{U}}(\cdot) := \widetilde{\mathcal{V}}(\cdot)/\sqrt{W(\cdot,\cdot)}$, it follows from Theorem 2 that

$$\mathcal{LR}_n^{(2)},\ \ \mathcal{LM}_n^{(2)} \Rightarrow \sup_{\gamma\in\Gamma} \min^2[0, \widetilde{\mathcal{U}}(\gamma)] \tag{3}$$

under the null. Therefore,

$$\mathbb{E}[\widetilde{\mathcal{U}}(\gamma_1)\widetilde{\mathcal{U}}(\gamma_2)] = \sum_{j=2}^{\infty}(\gamma_1\gamma_2)^j \frac{\Gamma(j+1/2)}{\Gamma(\frac{1}{2})\Gamma(j+1)} = \frac{1}{\sqrt{1-\gamma_1\gamma_2}} - 1 - \frac{1}{2}\gamma_1\gamma_2 = W(\gamma_1,\gamma_2)$$

by noting that

$$\sum_{j=0}^{\infty}(\gamma_1\gamma_2)^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(j+1)} = \frac{1}{\sqrt{1-\gamma_1\gamma_2}}$$

as shown by Lemma A.1 in the Appendix. This also implies that both $\mathcal{U}(\cdot)$ and $\widetilde{\mathcal{U}}(\cdot)$ have the same covariance kernel.

(b) The standard normal random variables $Z_2, Z_3, \ldots$ are obtained by applying the CLT to the generalized Laguerre polynomials in parallel to Theorem 1. That is, for each $j < \infty$,

$$-\frac{1}{\sqrt{n}} \sum_{t=1}^{n} L_j^{(-1/2)} \left(\frac{\widehat{Y}_t^2}{2} \right) \Rightarrow \sqrt{\frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(j+1)}} Z_j$$

by noting that for each j,

$$\int_0^{\infty} \left\{ L_j^{(-1/2)}(x) \right\}^2 \frac{1}{\sqrt{\pi}} \exp(-x) x^{-1/2} dx = \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(j+1)}$$

Here, $\pi^{-1/2}\exp(-(\cdot))(\cdot)^{-1/2}$ denotes the asymptotic probability density function of $\widehat{Y}_t^2/2$. The independence between Z_j and $Z_{j'}$ $(j \neq j')$ follows from the orthogonality of the generalized

Laguerre polynomials. That is, for $j \neq j'$,

$$\int_0^\infty L_j^{(-1/2)}(x)L_{j'}^{(-1/2)}(x)\frac{1}{\sqrt{\pi}}\exp(-x)x^{-1/2}dx = 0.$$

(c) The 2nd degree generalized Laguerre polynomial $L_2^{(-1/2)}(\cdot)$ is the first Laguerre polynomial used to rephrase the score function $D_n(\cdot)$ in Theorem 2 and is related to the kurtosis component of the LM test. We can see this feature by noting that the generalized Laguerre polynomial has a certain relationship with the Hermite polynomial. That is,

$$L_j^{(-1/2)}(x) = \frac{(-1)^j}{j!2^{2j}}H_{2j}(\sqrt{x}).$$

Therefore, it follows that

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} L_2^{(-1/2)}\left(\frac{\widehat{Y}_t^2}{2}\right) = -\sqrt{\frac{\Gamma(2.5)}{\Gamma(0.5)\Gamma(3)}}\left(\frac{H_{4,n}}{\sqrt{24n}}\right) \Rightarrow \sqrt{\frac{\Gamma(2.5)}{\Gamma(0.5)\Gamma(3)}}Z_4,$$

where $H_{4,n}/\sqrt{24n}$ is the kurtosis component used to define $\mathcal{LM}_n^{(1)}$. This fact implies that the test basis formed by the kurtosis component constitutes the test basis of both tests.

(d) We initially assumed a simple mixture of normal distributions, but this approach can be used to test for a mixture of conditional normals as well, according to Amengual et al. (2025, proposition 8). When testing for a mixture of conditional normals with two distinct scale parameters, the resultant Gaussian process emerges as $\mathcal{U}(\cdot)$.

3 Simulations

In this section, we conduct simulations and compare the LR and LM tests using the asymptotic critical values obtained from the two Gaussian versions.

3.1 Normal Mixture with Distinct Means

For the null simulation, we simulate $Y_t \sim \text{IID}\mathcal{N}(0,1)$ and test the hypothesis that Y_t follows a normal distribution. We set $\mu_{1*}, \mu_{2*} \in [-2,2]$ but do not restrict the parameter space for σ_*^2. Given that $\mu_* = 0$ under the null hypothesis, this specification implies that $\Delta = [-2,2]$. We obtain the null limit critical values by grid search and report them in Table A.1. These values are obtained by repeating 100,000 independent experiments using the formulas given in (1) and (2) for the LR and LM tests, respectively. Here, we have approximated $\widetilde{\mathcal{G}}(\cdot)$ by

$$\widetilde{\mathcal{G}}_k(\cdot) := \frac{1}{\sqrt{V(\cdot,\cdot)}} \sum_{j=3}^{k} \frac{(\cdot)^j}{\sqrt{j!}} Z_j$$

with $k = 500$. As k is large, the distributional difference between $\widetilde{\mathcal{G}}(\cdot)$ and $\widetilde{\mathcal{G}}_k(\cdot)$ is negligible.

Table A.2 reports the empirical rejection rates of the LR and LM tests under the null. The total number of experiments is 10,000. The empirical rejection rates of the LR test are similar to the nominal significance levels for each level. Meanwhile, the LM test exhibits a conservative testing result when the sample size is small, but it converges to the nominal levels as the sample size increases.

For the power simulation, we let

$$Y_t \sim \text{ IID} \begin{cases} \mathcal{N}(-1,1), & \text{w.p. } 1/2; \\ \mathcal{N}(1,1), & \text{w.p. } 1/2, \end{cases}$$

so that the alternative hypothesis is valid. Table A.3 reports the simulation results obtained by repeating 2,000 independent experiments. For both tests, the empirical rejection rates converge to 100% as the sample size increases. We also note that the LR test is consistently more powerful than the LM test at each significance level.

The simulation results suggest that the LR test is more beneficial than the LM test when testing for the mixture of normal distributions driven by two distinct means.

3.2 Normal Mixture with Distinct Variances

Using the Gaussian version $\widetilde{\mathcal{U}}(\cdot)$, we conduct another simulation to compare the performances of the LR and LM tests.

We first determine the asymptotic critical values for both tests in the simulation. We repeat 100,000 independent experiments using the formula given in (3). Here, we have approximated $\widetilde{\mathcal{G}}(\cdot)$ by

$$\widetilde{\mathcal{U}}_k(\cdot) := \frac{1}{\sqrt{W(\cdot,\cdot)}} \sum_{j=2}^{k} (\cdot)^j \sqrt{\frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(j+1)}} Z_j$$

with $k = 500$. We let σ_{1*}^2, $\sigma_{2*}^2 \in [1/2, 3/2]$ but do not restrict the parameter space for μ_*. As we set $\sigma_*^2 = 1$ under the null, $\Gamma = [-1/2, 1/2]$. We report the critical values in Table A.4. Both LR and LM tests have the same asymptotic critical values.

Table A.5 reports the empirical rejection rates of the LR and LM tests under the null. We let $Y_t \sim$ IID$\mathcal{N}(0,1)$ and test the normal distribution hypothesis. The total number of iterations is 10,000. The level performance is different from Section 3.1. The empirical rejection rates of the LM test are more similar to the nominal significance levels than the LR test. Meanwhile, the LR test is conservative when the sample size is small, although it converges to the nominal levels as the sample size grows.

For the power simulation, we let

$$Y_t \sim \text{ IID} \begin{cases} \mathcal{N}(0, 0.6), & \text{w.p. } 1/2; \\ \mathcal{N}(0, 1.4), & \text{w.p. } 1/2, \end{cases}$$

so that the alternative hypothesis is valid. Table A.6 reports the power simulation results. For both tests, the empirical rejection rates converge to 100% as the sample size increases. We also note that the LM test is always more powerful than the LR test for each significance level. This is opposite to that in Section 3.1.

The simulation results imply that the LM test is more useful than the LR test when testing for the mixture of normals driven by two different variances.

4 Concluding Remarks

The current study analytically derives the versions of the Gaussian processes associated with testing for the normal mixtures. The Gaussian versions are useful as we can exploit them to obtain the asymptotic critical values of the LR and LM tests by simulation. We examine two normal mixtures. One is the normal mixture with two different means and a single variance, and another is the normal mixture with two different variances and a single mean. For each model, we obtain the Gaussian version by associating the related score function with the Hermite or generalized Laguerre polynomial.

We compare the performances of the LR and LM tests using the Gaussian versions through simulation. Both models yield distinct simulation outcomes. When two different means are allowed, the LR test outperforms the LM test in terms of level and power. However, the LM test outperforms the LR test when two different variances are assumed. Choosing a suitable test tailored to a specific mixture hypothesis should lead to enhanced finite sample accuracy in testing for a normal mixture.

A Appendix

In the Appendix, we prove the main theorems in Section 2. Before proving them, we first provide a supplementary lemma.

Lemma A.1.

$$\sum_{j=0}^{\infty} s^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(j+1)} = \frac{1}{\sqrt{1-s}}.$$

Proof of Lemma A.1: We note that

$$\begin{aligned}\sum_{j=0}^{\infty} s^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(j+1)} &= \frac{1}{\Gamma(\frac{1}{2})}\sum_{j=1}^{\infty} s^{j-1}\frac{\Gamma(j-\frac{1}{2})}{\Gamma(j)} = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})}\sum_{j=1}^{\infty} s^{j-1}\frac{\Gamma(j-\frac{2}{2})}{(j-1)!(-\frac{1}{2})!} \\ &= \sum_{j=1}^{\infty} s^{j-1}\binom{j-\frac{3}{2}}{j-1} = \sum_{j=0}^{\infty} s^j \binom{j-\frac{1}{2}}{j} = \sum_{j=0}^{\infty} s^j \binom{-\frac{1}{2}}{j}(-1)^j \\ &= \sum_{j=0}^{\infty} (-s)^j \binom{-\frac{1}{2}}{j} = (1-s)^{-1/2} = \frac{1}{\sqrt{1-s}}.\end{aligned}$$

This completes the proof. ■

We now prove the main theorems.

Proof of Theorem 1: Using the formula of the Hermite polynomial generating function, we first note that

$$\exp\left[-\delta\widehat{Y}_t - \frac{\delta^2}{2}\right] = \sum_{j=0}^{\infty}\frac{1}{j!}H_j\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right)\left(-\frac{\delta}{\sqrt{2}}\right)^j,$$

where

$$H_0\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right)\left(-\frac{\delta}{\sqrt{2}}\right)^0 = 1, \quad H_1\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right)\left(-\frac{\delta}{\sqrt{2}}\right)^1 = -\delta\widehat{Y}_t, \quad H_2\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right)\left(-\frac{\delta}{\sqrt{2}}\right)^2 = (\widehat{Y}_t^2 - 1)\delta^2$$

by noting that $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) := 8x^3 - 12x$, and so on (e.g., Spiegel, 1968, p. 151), implying that

$$\sum_{j=0}^{2}\frac{1}{j!}H_j\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right)\left(-\frac{\delta}{\sqrt{2}}\right)^j = 1 - \delta\widehat{Y}_t + \frac{\delta^2}{2}(\widehat{Y}_t^2 - 1),$$

so that

$$G_n(\delta) := \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left\{ 1 - \delta \widehat{Y}_t + \frac{\delta^2}{2}(\widehat{Y}_t^2 - 1) - \exp\left[\frac{-\delta^2}{2} - \delta \widehat{Y}_t\right] \right\}$$
$$= - \sum_{j=3}^{\infty} \frac{1}{j!} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} H_j\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right)\left(-\frac{\delta}{\sqrt{2}}\right)^j.$$

We here note that if n is sufficiently large, $\widehat{Y}_t \overset{\text{A}}{\sim}$ IID $\mathcal{N}(0,1)$, so that if we apply the CLT, it further follows that for any $j < \infty$,

$$-\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \frac{1}{\sqrt{j!}} \left(-\frac{1}{\sqrt{2}}\right)^j H_j\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right) \Rightarrow Z_j \sim \mathcal{N}(0,1)$$

by noting that

$$\int_{-\infty}^{\infty} H_j\left(\frac{y}{\sqrt{2}}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} H_j^2\left(\frac{y}{\sqrt{2}}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy = j! 2^j$$

(e.g., Spiegel, 1968, p.152). Furthermore, for any $j \neq j'$,

$$\int_{-\infty}^{\infty} H_j\left(\frac{y}{\sqrt{2}}\right) H_{j'}\left(\frac{y}{\sqrt{2}}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy = 0$$

by the orthogonality of the Hermite polynomials (e.g., Spiegel, 1968, p. 152). Therefore, $\mathbb{E}[Z_j Z_{j'}] = 0$. From the normality, it implies that Z_j and $Z_{j'}$ are independent. Therefore,

$$G_n(\cdot) = \sum_{j=3}^{\infty} \frac{1}{\sqrt{j!}} \left[-\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \frac{1}{\sqrt{j!}} \left(-\frac{1}{\sqrt{2}}\right)^j H_j\left(\frac{\widehat{Y}_t}{\sqrt{2}}\right) \right] (\cdot)^j \Rightarrow \sum_{j=3}^{\infty} \frac{1}{\sqrt{j!}} Z_j (\cdot)^j.$$

This completes the proof. ■

Proof of Theorem 2: Using the formula of the generalized Laguerre polynomial generating function, we note that

$$\frac{1}{\sqrt{1-\gamma}} \exp\left[-\left(\frac{\gamma}{1-\gamma}\right) \frac{\widehat{Y}_t^2}{2}\right] = \sum_{j=0}^{\infty} \gamma^j L_j^{(-1/2)}\left(\frac{\widehat{Y}_t^2}{2}\right),$$

where

$$L_0^{(-1/2)}\left(\frac{\widehat{Y}_t^2}{2}\right) = 1 \quad \text{and} \quad L_1^{(-1/2)}\left(\frac{\widehat{Y}_t^2}{2}\right) = \frac{1}{2} - \frac{\widehat{Y}_t^2}{2}$$

by noting that $L_0^{(-1/2)}(x) = 1$, $L_1^{(-1/2)}(x) = \frac{1}{2} - x$, and so on (e.g., Hochstrasser, 1964, p. 779, 22.5.38), implying that

$$\sum_{j=0}^{1} \gamma^j L_j^{(-1/2)}\left(\frac{\widehat{Y}_t^2}{2}\right) = 1 + \frac{\gamma}{2}(1-\widehat{Y}_t^2),$$

so that

$$\begin{aligned} D_n(\gamma) &:= \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left\{1+\frac{\gamma}{2}(1-\widehat{Y}_t^2) - \frac{1}{\sqrt{1-\gamma}}\exp\left[-\left(\frac{\gamma}{1-\gamma}\right)\frac{\widehat{Y}_t^2}{2}\right]\right\} \\ &= -\sum_{j=2}^{\infty}\frac{\gamma^j}{\sqrt{n}}\sum_{t=1}^{n} L_j^{(-1/2)}\left(\frac{\widehat{Y}_t^2}{2}\right). \end{aligned}$$

We here note that if n is sufficiently large, $\widehat{Y}_t \overset{\text{A}}{\sim}$ IID $\mathcal{N}(0,1)$, so that $\widehat{Y}_t^2 \overset{\text{A}}{\sim}$ IID $\mathcal{X}_1^2$, and the asymptotic probability density function of $X_t := \widehat{Y}_t^2/2$ can be given as follows:

$$f(x) := \frac{x^{-1/2}}{\sqrt{\pi}}\exp(-x).$$

We further note that for each j,

$$\int_0^\infty L_j^{(-1/2)}(x) f(x) = 0 \quad \text{and} \quad \int_0^\infty \left\{L_j^{(-1/2)}(x)\right\}^2 f(x) = \frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(j+1)}$$

that is uniformly bounded by $\frac{1}{2}$ with respect to j (e.g., Hochstrasser, 1964, p.775, 22.2.12). Therefore, we can apply the CLT for each j, so that for each j,

$$-\frac{1}{\sqrt{n}}\sum_{t=1}^{n} L_j^{(-1/2)}\left(\frac{\widehat{Y}_t^2}{2}\right) \Rightarrow \sqrt{\frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(j+1)}}Z_j.$$

Furthermore, we note that for any $j \neq j'$,

$$\int_0^\infty L_j^{(-1/2)}(x) L_{j'}^{(-1/2)}(x) f(x) dx = 0$$

by the orthogonality of the generalized Laguerre polynomials (e.g., Hochstrasser, 1964, pp. 773-775, 22.1.1 and 22.2.12), implying that $\mathbb{E}[Z_j Z_{j'}] = 0$, meaning that Z_j and $Z_{j'}$ are independent. Therefore,

$$D_n(\cdot) = \sum_{j=2}^{\infty}(\cdot)^j\left[-\frac{1}{\sqrt{n}}\sum_{t=1}^{n} L_j^{(-1/2)}\left(\frac{\widehat{Y}_t^2}{2}\right)\right] \Rightarrow \sum_{j=2}^{\infty}(\cdot)^j\sqrt{\frac{\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(j+1)}}Z_j.$$

This completes the proof. ■

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Test \ Level	1.00%	2.50%	5.00%	7.50%	10.00%
$\mathcal{LR}_n^{(1)}$	8.69	6.91	5.63	4.86	4.31
$\mathcal{LM}_n^{(1)}$	9.23	7.38	6.07	5.27	4.69

Table A.1: ASYMPTOTIC CRITICAL VALUES OF THE LR AND LM TESTS. Figures show the asymptotic critical values of the LR and LM tests obtained by applying the grid search method to the practical Gaussian process $\widetilde{\mathcal{G}}(\cdot)$. The parameter space of μ_* is $[-2.0, 2.0]$. The critical values are obtained by repeating independent experiments 100,000 times.

Test	Level \ n	50	100	200	300	400	500
$\mathcal{LR}_n^{(1)}$	1.00%	1.33	1.06	1.00	1.02	1.02	1.05
	2.50%	3.09	2.83	2.40	2.58	2.62	2.31
	5.00%	5.69	5.47	4.79	5.23	5.02	4.66
	7.50%	8.11	7.67	7.32	7.74	7.20	6.99
	10.0%	10.76	10.10	9.73	10.18	9.73	9.63
$\mathcal{LM}_n^{(1)}$	1.00%	1.78	1.88	2.20	2.32	2.73	2.68
	2.50%	2.45	2.86	3.10	3.43	3.88	3.90
	5.00%	3.25	3.88	4.38	4.90	5.47	5.49
	7.50%	4.02	4.98	5.63	6.28	7.05	7.04
	10.0%	4.94	6.07	7.14	7.63	8.82	8.98

Table A.2: EMPIRICAL REJECTION RATES UNDER THE NULL (IN PERCENT). Figures show the empirical rejection rates of the LR and LM tests. DGP: $Y_t \sim$ IID$\mathcal{N}(0, 1)$. The parameter space of μ_* is $[-2.0, 2.0]$. The total number of replications is 10,000.

Test	Level \ n	100	400	700	1,000	1,500	2,000
$\mathcal{LR}_n^{(1)}$	1.00%	5.80	54.05	84.50	95.65	99.85	100.0
	2.50%	12.85	67.10	91.35	98.00	99.95	100.0
	5.00%	21.00	76.60	95.10	98.90	99.95	100.0
	7.50%	28.30	82.20	96.70	99.20	99.95	100.0
	10.0%	34.75	82.85	97.30	99.50	99.95	100.0
$\mathcal{LM}_n^{(1)}$	1.00%	0.15	4.95	37.25	72.40	93.55	99.55
	2.50%	0.25	18.60	59.40	82.85	97.80	99.85
	5.00%	0.55	34.25	73.80	92.50	99.00	99.95
	7.50%	1.15	46.80	81.25	95.40	99.35	99.95
	10.0%	2.80	56.25	86.45	97.15	99.60	99.95

Table A.3: EMPIRICAL REJECTION RATES UNDER THE ALTERNATIVE (IN PERCENT). Figures show the empirical rejection rates of the LR and LM tests. DGP: $Y_t \sim$ IID$\mathcal{N}(-1, 1)$ with probability 1/2; and $\mathcal{N}(1, 1)$ with probability 1/2. The parameter space of μ_* is $[-2.0, 2.0]$. The total number of replications is 2,000.

Test \ Level	1.00%	2.50%	5.00%	7.50%	10.00%
$\mathcal{LR}_n^{(2)}$	6.64	4.94	3.69	2.97	2.47
$\mathcal{LM}_n^{(2)}$	6.64	4.94	3.69	2.97	2.47

Table A.4: ASYMPTOTIC CRITICAL VALUES OF THE LR AND LM TESTS. Figures show the asymptotic critical values of the LR and LM tests obtained by applying the grid search method to the practical Gaussian process $\widetilde{\mathcal{U}}(\cdot)$. The parameter space of σ_*^2 is $[1/2, 3/2]$. The critical values are obtained by repeating independent experiments 100,000 times.

Test	Level \ n	1,000	5,000	10,000	20,000	40,000	60,000
$\mathcal{LR}_n^{(2)}$	1.00%	0.59	0.56	0.84	1.01	0.81	0.92
	2.50%	1.69	1.75	2.20	2.44	2.02	2.35
	5.00%	3.28	3.80	4.40	4.74	4.31	4.53
	7.50%	5.28	5.94	6.51	6.85	6.51	6.91
	10.0%	7.25	8.02	8.72	9.14	8.77	9.44
$\mathcal{LM}_n^{(2)}$	1.00%	2.22	1.92	1.83	1.95	1.68	1.61
	2.50%	3.80	3.63	3.40	3.60	3.13	3.15
	5.00%	6.08	5.93	5.90	6.28	5.77	5.93
	7.50%	8.14	8.15	8.35	9.08	8.37	8.49
	10.0%	10.05	10.45	10.51	11.40	10.78	10.73

Table A.5: EMPIRICAL REJECTION RATES UNDER THE NULL (IN PERCENT). Figures show the empirical rejection rates of the LR and LM tests. DGP: $Y_t \sim$ IID$\mathcal{N}(0, 1)$. The parameter space of σ_*^2 is $[1/2, 3/2]$. The total number of replications is 10,000.

Test	Level \ n	200	500	800	1,000	1,500	2,000
$\mathcal{LR}_n^{(2)}$	1.00%	4.50	17.90	45.65	54.85	76.65	88.75
	2.50%	9.80	30.00	59.85	68.40	85.35	94.50
	5.00%	17.00	40.00	71.45	78.65	91.25	97.00
	7.50%	22.80	47.20	78.00	84.30	93.85	98.15
	10.0%	28.40	52.55	81.70	87.40	95.35	98.75
$\mathcal{LM}_n^{(2)}$	1.00%	18.80	38.60	57.70	68.65	82.70	92.55
	2.50%	26.25	49.35	67.05	77.70	89.40	96.55
	5.00%	33.15	58.65	75.25	83.60	93.10	97.70
	7.50%	38.30	64.85	79.70	87.10	95.65	98.20
	10.0%	42.35	68.65	82.45	89.75	96.75	98.65

Table A.6: EMPIRICAL REJECTION RATES UNDER THE ALTERNATIVE (IN PERCENT). Figures show the empirical rejection rates of the LR and LM tests. DGP: $Y_t \sim$ IID$\mathcal{N}(0, 0.6)$ with probability 1/2; and $\mathcal{N}(0, 1.4)$ with probability 1/2. The parameter space of σ_*^2 is $[1/2, 3/2]$. The total number of replications is 2,000.