Online Supplement to "Sequentially Testing Polynomial Model Hypotheses using Power Transforms of Regressors"

JIN SEO CHO
School of Economics
Yonsei University, 50 Yonsei-ro
Seodaemun-gu, Seoul, 120-749, Korea

PETER C.B. PHILLIPS
Yale University, University of Auckland
Singapore Management University &
University of Southampton

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Abstract

This supplement provides proofs of the subsidiary lemmas and the main results in the text of "Sequentially Testing Polynomial Model Hypotheses using Power Transforms of Regressors" by J. S. Cho and P. C. B Phillips (2017).

1 Preliminary Lemmas and Proofs

1.1 Claims

Lemma A1. *Given Assumptions 1 and 2,*

(i) $A_c'U = O_{\mathbb{P}}(\sqrt{n})$, $Z'U = O_{\mathbb{P}}(\sqrt{n})$, *and* $E_c'U = O_{\mathbb{P}}(\sqrt{n})$, *where* $E_c := (d/d\gamma)Q_c(c) = [0_{n\times c} \vdots A_c \vdots 0_{n\times(m-c+k)}]$, $Q_c(\gamma) := [X(0), \ldots, X(c-1), X(\gamma), X(c+1), \ldots, X(m), D]$, *and* $D := [d_1, \ldots, d_n]'$*;*

(ii) $A_c'Z = O_{\mathbb{P}}(n)$, $Z'Z = O_{\mathbb{P}}(n)$, *and* $E_c'Z = O_{\mathbb{P}}(n)$*;*

(iii) $A_c'A_c = O_{\mathbb{P}}(n)$, $A_c'E_c = O_{\mathbb{P}}(n)$, $B_c'U = O_{\mathbb{P}}(n)$, $B_c'Z = O_{\mathbb{P}}(n)$, $E_c'Z = O_{\mathbb{P}}(n)$, $E_c'E_c = O_{\mathbb{P}}(n)$, $F_c'U = O_{\mathbb{P}}(n)$, *and* $F_c'Z = O_{\mathbb{P}}(n)$, *where* $F_c := (d^2/d\gamma^2)Q_c(c) = [0_{n\times c} \vdots B_c \vdots 0_{n\times(m-c+k)}]$*; and*

(iv) $B_c'U = o_{\mathbb{P}}(n)$ *and* $F_c'U = o_{\mathbb{P}}(n)$. □

Lemma A2. *Given Assumptions 1 and 2,*

(i) if $\mathbb{E}[y_t|x_t, d_t] = x_t(m)'\alpha_* + d_t'\eta_* + s(x_t)$ *with* $\mathbb{E}[s(x_t)^2] < \infty$ *and* $\mathbb{E}[\log^{4j_*}(x_t)] < \infty$*, for some* $\widetilde{\gamma} \in \Gamma$, $h(\widetilde{\gamma}) \in (0, h_0)$ *and*

$$\frac{1}{n}QLR_n = \left(1 - \frac{h(\widetilde{\gamma})}{h_0}\right) + o_{\mathbb{P}}(1),$$

where $j_* := \min\{j \in \mathbb{N} : \mathbb{E}[v_t \log^j(x_t)] \neq 0\}$*, and* v_t *is the linear projection error obtained by projecting* y_t *into the space of* $(x_t(m)', d_t')'$*;*

(ii) if $\mathbb{E}[y_t|x_t, d_t] = x_t(m)'\alpha_* + d_t'\eta_* + n^{-1/2}s(x_t)$ *with* $|s(x_t)| \leq m_t$*,*

$$QLR_n \Rightarrow \sup_{\gamma\in\Gamma}\left(\mathcal{Z}(\gamma) + \frac{\zeta(\gamma)}{\sigma(\gamma)}\right)^2,$$

where $\zeta(\gamma) := \mathbb{E}[s(x_t)x_t^{\gamma}] - \mathbb{E}[s(x_t)z_t']E[z_tz_t']^{-1}\mathbb{E}[z_tx_t^{\gamma}]$. □

Lemma A3. *Given Assumptions 3 and 4,*

(i) if for some $m_0 > m$*,* $\mathbb{E}[y_t|d_t] = t(m_0)'\alpha_* + d_t'\eta_*$*,*

$$\frac{1}{n}QLR_n = \sup_{\gamma\in\Gamma}\frac{\widetilde{\sigma}^2(\gamma, m_0)}{\{\widetilde{\sigma}^2(\gamma,\gamma)\}^{1/2}\{\widetilde{\sigma}^2(m_0, m_0)\}^{1/2}} + o_{\mathbb{P}}(1);$$

(ii) if $\mathbb{E}[y_t|d_t] = t(m)'\alpha_* + d_t'\eta_* + s(t)$ *with* $s(\cdot)$ *being a smoothly slowly varying (SSV) function as in Phillips (2007), and* $ns'(n) \to c$ $(\neq 0)$*,*

$$\frac{1}{n}QLR_n = \sup_{\gamma\in\Gamma}\left(\frac{c^2\sigma_*^2}{\sigma_*^2 + c^2q}\right)\left(\frac{p(\gamma)}{\widetilde{\sigma}(\gamma,\gamma)}\right)^2 + o_{\mathbb{P}}(1),$$

where $p(\gamma) := (\gamma-1)(7\gamma+15)/\{4(\gamma+1)^2(\gamma+2)\}$ *and* $q := 91/64$*;*

(iii) if $\mathbb{E}[y_t|d_t] = t(m)'\alpha_* + d_t'\eta_* + s(t)$ *with* $s(\cdot)$ *being an SSV function, and* $ns'(n) \to \infty$*,*

$$\frac{1}{n}QLR_n = \sup_{\gamma\in\Gamma}\left(\frac{\sigma_*^2}{q}\right)\left(\frac{p(\gamma)}{\widetilde{\sigma}(\gamma,\gamma)}\right)^2 + o_{\mathbb{P}}(1),$$

(iv) if $\mathbb{E}[y_t|d_t] = t(m)'\alpha_* + d_t'\eta_* + s(t)/\{n^{3/2}s'(n)\}$ *with* $s(\cdot)$ *being an SSV function,*

$$QLR_n \Rightarrow \sup_{\gamma\in\Gamma}\left(\widetilde{\mathcal{Z}}(\gamma) + \frac{p(\gamma)}{\widetilde{\sigma}(\gamma)}\right)^2.$$ □

2 Proofs of the Preliminary Lemmas

Proof of Lemma A1: (*i*) By the definition of $E_c := [0_{n\times c} \vdots A_c \vdots 0_{n\times(m-c+k)}]$, we note that if $A_c'U = O_{\mathbb{P}}(\sqrt{n})$, then $E_c'U = O_{\mathbb{P}}(\sqrt{n})$. Therefore, we focus on proving that $A_c'U = O_{\mathbb{P}}(\sqrt{n})$.

By the definition of $A_c'U$, $n^{-1/2}A_c'U = [n^{-1/2}\sum x_t^c \log(x_t)u_t]$, so that if $\mathbb{E}[x_t^{2c}\log^2(x_t)\, u_t^2] < \infty$, we can apply the martingale central limit theorem (CLT). Using the Cauchy-Schwarz inequality, we obtain: (*a*) $\mathbb{E}[x_t^{2c}\log^2(x_t)u_t^2] \leq \mathbb{E}[x_t^{4c}\log^4(x_t)]^{1/2}\mathbb{E}[u_t^4]^{1/2} \leq \mathbb{E}[x_t^{8c}]^{1/4}\mathbb{E}[\log^8(x_t)]^{1/4}\mathbb{E}[u_t^4]^{1/2}$; (*b*) $\mathbb{E}[x_t^{2c}\log^2(x_t)u_t^2]$

$\leq \mathbb{E}[u_t^4 \log^4(x_t)]^{1/2}\mathbb{E}[x_t^{4c}]^{1/2} \leq \mathbb{E}[u_t^8]^{1/4}\mathbb{E}[\log^8(x_t)]^{1/4}\mathbb{E}[x_t^{4c}]^{1/2}$; and (*c*) $\mathbb{E}[x_t^{2c}\log^2(x_t)u_t^2] \leq \mathbb{E}[x_t^{4c}u_t^4]^{1/2}$ $\mathbb{E}[\log^4(x_t)]^{1/2} \leq \mathbb{E}[x_t^{8c}]^{1/4}\ \mathbb{E}[u_t^8]^{1/4}\mathbb{E}[\log^4(x_t)]^{1/2}$. We now note that the elements in the right side of (*a*), (*b*), and (*c*) are finite by Assumption 2(*iii*), respectively.

As for $Z'U$, $n^{-1/2}Z'U = n^{-1/2}\sum z_{t,i}u_t$ obeys a CLT if $\mathbb{E}[z_{t,i}^2 u_t^2] < \infty$. We note that $\mathbb{E}[z_{t,i}^2 u_t^2] \leq \mathbb{E}[z_{t,i}^4]^{1/2}\mathbb{E}[u_t^4]^{1/2}$ by the Cauchy-Schwarz inequality. If $\mathbb{E}[z_{t,i}^4] < \infty$ and $\mathbb{E}[u_t^4] < \infty$, the desired results follow. These conditions are already required in Assumption 2.

(*ii*) As in (*i*), if $A_c'Z = O_{\mathbb{P}}(n)$, $E_c'Z = O_{\mathbb{P}}(n)$ by the definition of E_c. For $A_c'Z = [\sum x_t^c \log\,(x_t)z_{t,i}]$, this obeys the law of large numbers (LLN) if $\mathbb{E}[|x_t^c\log(x_t)z_{t,i}|] < \infty$. We consider two cases separately: for some ℓ, when $z_{t,i} = d_{t,\ell}$ and when $z_{t,i} = x_t^\ell$.

Take the case: $z_{t,i} = d_{t,\ell}$. Note that $\mathbb{E}[x_t^c\log(x_t)z_{t,i}] = \mathbb{E}[x_t^c\log(x_t)d_{t,\ell}]$. Therefore, (*a*) $\mathbb{E}[x_t^c\log(x_t)d_{t,\ell}] \leq \mathbb{E}[x_t^{2c}\log^2(x_t)]]^{1/2}\mathbb{E}[d_{t,\ell}^2]^{1/2} \leq \mathbb{E}[x_t^{4c}]^{1/4}\mathbb{E}[\log^4(x_t)]^{1/4}\mathbb{E}[d_{t,\ell}^2]^{1/2}$; (*b*) $\mathbb{E}[x_t^c\log\,(x_t)d_{t,\ell}] \leq \mathbb{E}[d_{t,\ell}^2\log^2(x_t)]^{1/2}\mathbb{E}[x_t^{2c}]^{1/2} \leq \mathbb{E}[d_{t,\ell}^4]^{1/4}\mathbb{E}[\log^4(x_t)]^{1/4}\mathbb{E}[x_t^{2c}]^{1/2}$; (*c*) $\mathbb{E}[x_t^c\log(x_t)d_{t,\ell}] \leq \mathbb{E}[x_t^{2c}d_{t,\ell}^2]^{1/2}\mathbb{E}[\log^2(x_t)]^{1/2} \leq \mathbb{E}[x_t^{4c}]^{1/4}\mathbb{E}[d_{t,\ell}^4]^{1/4}\mathbb{E}[\log^2(x_t)]^{1/2}$ by the Cauchy-Schwarz inequality. All these bounds are finite by Assumption 2(*iii*).

Next consider the case when $z_{t,i} = x_t^\ell$. Then, $\mathbb{E}[x_t^c\log(x_t)z_{t,i}] = \mathbb{E}[x_t^{c+\ell}\log(x_t)]$, which is bounded by $\mathbb{E}[x_t^{2(c+\ell)}]^{1/2}\mathbb{E}[\log^2(x_t)]^{1/2}$. We note that Assumption 2(*iii*) then ensures the required finite bound.

As for $Z'Z$, $n^{-1}Z'Z = n^{-1}\sum z_{t,i}z_{t,\ell}$ obeys an LLN if $\mathbb{E}[|z_{t,i}z_{t,\ell}|] < \infty$. We note that $\mathbb{E}[|z_{t,i}z_{t,\ell}|] \leq \mathbb{E}[z_{t,i}^2]^{1/2}\mathbb{E}[z_{t,\ell}^2]^{1/2}$ by the Cauchy-Schwarz inequality. If $\mathbb{E}[z_{t,i}^2] < \infty$, the desired results follows as it is assumed in Assumption 2(*iii*).

(*iii*) By the definitions of E_c and $F_c := [0_{n\times c} \vdots B_c \vdots 0_{n\times(m-c+k)}]$, if $A_c'A_c = O_{\mathbb{P}}(n)$, $B_c'U = O_{\mathbb{P}}(n)$, $B_c'Z = O_{\mathbb{P}}(n)$, and $A_c'Z = O_{\mathbb{P}}(n)$ then $A_c'E_c = O_{\mathbb{P}}(n)$, $F_c'U = O_{\mathbb{P}}(n)$, $F_c'Z = O_{\mathbb{P}}(n)$, $E_c'E_c = O_{\mathbb{P}}(n)$, and $E_c'Z = O_{\mathbb{P}}(n)$. We have already shown that $A_c'Z = O_{\mathbb{P}}(n)$ in (*ii*). We, therefore, focus on proving $A_c'A_c = O_{\mathbb{P}}(n)$, $B_c'U = O_{\mathbb{P}}(n)$, and $B_c'Z = O_{\mathbb{P}}(n)$.

We examine each case in turn. (*a*) We note that $n^{-1}A_c'A_c = n^{-1}\sum x_t^{2c}\log^2(x_t)$, so that if $\mathbb{E}[x_t^{2c}\log^2(x_t)] < \infty$, the LLN holds. We note that $\mathbb{E}[x_t^{2c}\log^2(x_t)] \leq \mathbb{E}[x_t^{4c}]^{1/2}\mathbb{E}[\log^4(x_t)]^{1/2}$, and the right side is finite by Assumption 2(*iii*).

(*b*) Note that $n^{-1}B_c'U = n^{-1}\sum x_t^c\log^2(x_t)u_t$ and, if $\mathbb{E}[|x_t^c\log^2(x_t)u_t|] < \infty$, the LLN holds. We also note that (*b.i*) $\mathbb{E}[x_t^c\log^2(x_t)u_t] \leq \mathbb{E}[x_t^{2c}\log^4(x_t)]^{1/2}\ \mathbb{E}[u_t^2]^{1/2} \leq \mathbb{E}[x_t^{4c}]^{1/4}\mathbb{E}[\log^8(x_t)]^{1/4}\ \mathbb{E}[u_t^2]^{1/2}$; (*b.ii*) $\mathbb{E}[x_t^c\log^2(x_t)u_t] \leq \mathbb{E}[u_t^2\log^4(x_t)]^{1/2}\mathbb{E}[x_t^{2c}]^{1/2} \leq \mathbb{E}[u_t^4]^{1/4}\mathbb{E}[\log^8(x_t)]^{1/4}\mathbb{E}[x_t^{2c}\ ]^{1/2}$; and (*b.iii*) $\mathbb{E}[x_t^c\log^2(x_t)u_t] \leq \mathbb{E}[u_t^2x_t^{2c}]^{1/2}\mathbb{E}[\log^2(x_t)]^{1/2} \leq \mathbb{E}[u_t^4]^{1/4}\mathbb{E}[x_t^{4c}]^{1/4}\mathbb{E}[\log^2(x_t)]^{1/2}$. Thus, each of the elements forming the right side is finite by Assumption 2(*ii.a*), 2(*ii.b*), and 2(*ii.c*), respectively.

(*c*) Finally, we examine $n^{-1}B_c'Z = [n^{-1}\sum x_t^c\log^2(x_t)z_{t,i}]$. As before, there are two separate cases:

for some ℓ, $z_{t,i} = d_{t,\ell}$ or $z_{t,i} = x_t^\ell$. We first consider $z_{t,i} = d_{t,\ell}$. Note that $\mathbb{E}[|x_t^c \log^2(x_t) z_{t,i}|] = \mathbb{E}[|x_t^c \log^2 (x_t) d_{t,\ell}|]$. Therefore, (*c.i*) $\mathbb{E}[|x_t^c \log^2(x_t) d_{t,\ell}|] \leq \mathbb{E}[x_t^{2c} \log^4(x_t)]^{1/2} \mathbb{E}[d_{t,\ell}^2]^{1/2} \leq \mathbb{E}[x_t^{4c}]^{1/4} \mathbb{E}[\log^8(x_t)]^{1/4} \mathbb{E}[d_{t,\ell}^2]^{1/2}$; (*c.ii*) $\mathbb{E}[x_t^c \log^2(x_t) x_{t,i}] \leq \mathbb{E}[d_{t,\ell}^2 \log^4(x_t)]^{1/2} \mathbb{E}[x_t^{2c}]^{1/2} \leq \mathbb{E}[d_{t,\ell}^4]^{1/4} \mathbb{E}[\log^8(x_t)]^{1/4} \mathbb{E}[x_t^{2c}]^{1/2}$; and (*c.iii*) $\mathbb{E}[|x_t^c \log^2(x_t) d_{t,\ell}|] \leq \mathbb{E}[d_{t,\ell}^2 \, x_t^{2c}]^{1/2} \mathbb{E}[\log^4(x_t)]^{1/2} \leq \mathbb{E}[d_{t,\ell}^4]^{1/4} \, \mathbb{E}[x_t^{4c}]^{1/4} \mathbb{E}[\log^4(x_t)]^{1/2}$. Then, the right sides are finite by Assumption 2(*iii.a*), 2(*iii.b*), and 2(*iii.c*), respectively.

Next consider $z_{t,i} = x_t^\ell$. Then, $\mathbb{E}[|x_t^c \log^2(x_t) z_{t,i}|] = \mathbb{E}[|x_t^{c+\ell} \log^2(x_t)|] \leq \mathbb{E}[|x_t^{2c+2\ell}|]^{1/2} \, \mathbb{E}[| \log^4(x_t) |]^{1/2}$. This bound is also finite by Assumption 2(*iii*).

(*iv*) By the definition of F_c, if $B_c'U = o_{\mathbb{P}}(n)$, it follows that $F_c'U = o_{\mathbb{P}}(n)$. We already proved that $B_c'U = O_{\mathbb{P}}(n)$ in (*iii*), and applying the LLN and the martingale difference sequence (MDS) condition in Assumption 2(*ii*) implies that $B_c'U = o_{\mathbb{P}}(n)$. This completes the proof. ■

Proof of Lemma A2: (*i* and *ii*) Assumptions 1 and 2 satisfy the regularity assumptions 1, 2(*iii*, *v*), 4(*ii*), and 5 of Baek, Cho, and Phillips (2015, BCP). Furthermore, we can let $[x_t, x_t^2, \ldots, x_t^m]$ be a part of d_t of BCP. From these two facts, the assumptions in theorem 5 of BCP are satisfied. Therefore, the BCP results apply to Lemma A2 with $m(x_t)$ of BCP being $s(x_t)$ in the current paper. ■

Proof of Lemma A3: **Part (i):** Given that $m_0 > m$, if we define $G(m_0) := \sum_{j=m+1}^{m_0} \alpha_{j*} [1^j, 2^j, \ldots, t^j, \ldots, (n-1)^j, n^j]'$, then

$$\widehat{\sigma}_{n,0}^2 - \widehat{\sigma}_{n,A}^2 = \sup_{\gamma \in \Gamma} \frac{\{n^{-1}(U + G(m_0))' M S(\gamma)\}^2}{(n^{-1} S(\gamma)' M S(\gamma))}.$$

Here, we note that $\sup_\gamma |n^{-1} U' M S(\gamma)| = o_{\mathbb{P}}(1)$. Furthermore, $G(m_0) = O(n^{m_0})$ and $n^{-m_0} G(m_0) = \alpha_{m_0 *} S(m_0) + o(1)$, so that $n^{-1} G(m_0)' M S(\gamma) = \alpha_{m_0 *} n^{m_0 - 1} S(m_0)' M S(\gamma) + O_{\mathbb{P}}(n^{m_0 - 2})$. This implies that $\sup_{\gamma \in \Gamma} |n^{-1-m_0} G(m_0)' M S(\gamma) - \alpha_{m_0 *} n^{-1} S(m_0)' M S(\gamma)| = o_{\mathbb{P}}(1)$, so it follows that

$$\widehat{\sigma}_{n,0}^2 - \widehat{\sigma}_{n,A}^2 = \sup_{\gamma \in \Gamma} \alpha_{m_0 *}^2 n^{2m_0} \frac{\{n^{-1} S(m_0)' M S(\gamma)\}^2}{(n^{-1} S(\gamma)' M S(\gamma))} + o_{\mathbb{P}}(n^{2m_0}). \tag{1}$$

We next note that $\widehat{\sigma}_{n,0}^2 = n^{-1}(U + G(m_0))' M (U + G(m_0))$. Hence,

$$\widehat{\sigma}_{n,0}^2 = \sigma_*^2 + \alpha_{m_0 *}^2 n^{2m_0} n^{-1} S(m_0)' M S(m_0) + o_{\mathbb{P}}(n^{2m_0}). \tag{2}$$

With these results in hand, (1) and (2) imply that

$$\frac{1}{n}QLR_n = \frac{\widehat{\sigma}^2_{n,0} - \widehat{\sigma}^2_{n,A}}{\widehat{\sigma}^2_{n,0}} = \sup_{\gamma\in\Gamma} \frac{\{n^{-1}S(\gamma)'MS(m_0)\}^2}{(n^{-1}S(\gamma)'MS(\gamma))(n^{-1}S(m_0)'MS(m_0))} + o_{\mathbb{P}}(1)$$
$$= \sup_{\gamma\in\Gamma} \frac{\widetilde{\sigma}^2(\gamma, m_0)}{\{\widetilde{\sigma}^2(\gamma,\gamma)\}^{1/2}\{\widetilde{\sigma}^2(m_0,m_0)\}^{1/2}} + o_{\mathbb{P}}(1),$$

by noting that $\widetilde{\sigma}^2(\cdot,\cdot)$ is the almost sure limit of $n^{-1}\widehat{\sigma}^2_{n,0}S(\cdot)'MS(\cdot)$.

Parts (ii, iii, and iv): In our context, we can let $\sigma^2_* g(\gamma,\widetilde{\gamma})$ and K of theorem 6 in BCP be $\widetilde{\sigma}(\gamma,\widetilde{\gamma})$ and 1, respectively. The desired results then follow from theorem 6(*ii.a*, *ii.b*, *v*). ■

3 Proofs of the Main Claims

Proof of Lemma 1: (*i*) To show the stated claim, we first derive the first-order derivative of $L_n(\gamma;\alpha_c)$ with respect to γ. Note that

$$L^{(1)}_n(\gamma;\alpha_c) = 2P_c(\alpha_c)'Q_c(\gamma)(Q_c(\gamma)'Q_c(\gamma))^{-1}\left[(d/d\gamma)Q_c(\gamma)'P_c(\alpha_c)\right]$$
$$+ P_c(\alpha_c)'Q_c(\gamma)\{(d/d\gamma)(Q_c(\gamma)'Q_c(\gamma))^{-1}\}Q_c(\gamma)'P_c(\alpha_c),$$

$Q_c(c) = Z$ from $Q_c(\gamma) := [X(0),\ldots,X(c-1),X(\gamma),X(c+1),\ldots,X(m),D]$ and $(d/d\gamma)Q_c(\gamma) = E_c$. Next, $P_c(\alpha_c) = Y - \alpha_c X(c) = Z[\alpha_{0*},\ldots,\alpha_{c-1},(\alpha_{c*}-\alpha_c),\alpha_{c+1},\ldots,\alpha_{m*},\eta_*']' + Z'U = Z\kappa_c + U$, so that $P_c(\alpha_c) = Z\kappa_c + U$, where $\kappa_c := [\alpha_{0*},\ldots,\alpha_{(c-1)*},(\alpha_{c*}-\alpha_c),\alpha_{(c+1)*},\ldots,\alpha_{m*},\eta_*']'$. Finally, we obtain that

$$(d/d\gamma)(Q_c(\gamma)'Q_c(\gamma))^{-1}_{\gamma=c} = -(Z'Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1} \quad (3)$$

and collect all these separate derivations in $(d/d\gamma)L_n(\gamma;\alpha_c)$. This yields that

$$L^{(1)}_n(c;\alpha_c) = 2(Z\kappa_c+U)'Z(Z'Z)^{-1}E_c'(Z\kappa_c+U)$$
$$-(Z\kappa_c+U)'Z(Z'Z)^{-1}(Z'E_c+E_c'Z)(Z'Z)^{-1}Z'(Z\kappa_c+U).$$

We further rearrange the terms on the right side. The first component is the sum of four other components : (a) $2\kappa_c'Z'Z(Z'Z)^{-1}E_c'Z\kappa_c = 2\kappa_c'E_c'Z\kappa_c$; (b) $2\kappa_c'E_c'U$; (c) $2U'Z(Z'Z)^{-1}E_c'Z\kappa_{cy} = 2\kappa_c'Z'E_c(Z'Z)^{-1}Z'$ U; and (d) $2U'Z(Z'Z)^{-1}E_c'U$. Next, the second component is the sum of four components: (a) $-\kappa_c'Z'E_c\kappa_c$ $-\kappa_c'E_c'Z\kappa_c = -2\kappa_c'E_c'Z\ \kappa_c$; (b) $-U'Z(Z'Z)^{-1}Z'E_c\,\kappa_c - \kappa_c'E_c'Z(Z'Z)^{-1}Z'U = -2\kappa_c'E_c'Z(Z'Z)^{-1}Z'\,U$; (c) $-U'Z(Z'Z)^{-1}E_c'Z\kappa_c - \kappa_c'Z'\ E_c(Z'Z)^{-1}Z'U = -2\kappa_c'Z'E_c(Z'Z)^{-1}Z'U$; and (d) $-U'Z(Z'Z)^{-1}(Z'E_c$

$+E_c'Z)(Z'Z)^{-1}Z'U$. If we collect these eight different components according to their order of convergence, they can be classified into the following three different terms:

- (a) $2\kappa_c' E_c' Z\kappa_c - 2\kappa_c' E_c' Z\kappa_c = 0$;
- (b, c) $2\kappa_c'\{E_c' + Z'E_c(Z'Z)^{-1}Z' - E_c'Z(Z'Z)^{-1}Z' - Z'E_c(Z'Z)^{-1}Z'\}U = 2(\alpha_{c*} - \alpha_c)A_c'MU$ because $Z'E_c = A_c'$;
- (d) $2U'Z(Z'Z)^{-1}E_c'U - U'Z(Z'Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1}Z'U$,

so that the first-order derivative is now obtained as

$$L_n^{(1)}(c;\alpha_c) = 2(\alpha_{c*} - \alpha_c)A_c'MU + 2U'E_c(Z'Z)^{-1}Z'U - U'Z(Z'Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1}Z'U,$$

and this is the desired first-order derivative. Given this derivative, Lemma A1(*i* and *ii*) implies that the second and third terms in the right side are $o_{\mathbb{P}}(n)$, so that the desired result follows from this.

(*iii*) We next examine the second-order derivative. In the same way, we obtain that

$$\begin{aligned} L_n^{(2)}(c;\alpha_c) =& 2(P_c(\alpha_c)'E_c)(Z'Z)^{-1}(E_c'P_c(\alpha_c)) + 4(P_c(\alpha_c)'Z)\{(d/d\gamma)[Q_c c'Q_c c]^{-1}\}E_c'P_c(\alpha_c) \\ &+ 2(P_c(\alpha_c)'Z)(Z'Z)^{-1}F_c'P_c(\alpha_c) + (P_c(\alpha_c)'Z)\{(d^2/d\gamma^2)[Q_c c'Q_c c]^{-1}\}Z'P_c(\alpha_c). \end{aligned}$$

We note that (3) already provides the form of $(d/d\gamma)[Q(\gamma)'Q(\gamma)]^{-1}_{\gamma=c}$, and

$$\begin{aligned} (d^2/d\gamma^2)[Q(\gamma)'Q(\gamma)]^{-1}_{\gamma=c} =& 2Z(Z'Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1}Z' \\ &- (Z'Z)^{-1}(2E_c'E_c + Z'F_c + F_c'Z)(Z'Z)^{-1}. \end{aligned}$$

Using these and the previous definitions, the second-order derivative is obtained as

$$\begin{aligned} L_n^{(2)}(c;\alpha_c) = & 2(Z\kappa_c + U)'\{E_c(Z'Z)^{-1}E_c' + Z(Z'Z)^{-1}F_c'\}(Z\kappa_c + U) \\ &- 4(Z\kappa_c + U)'Z(Z'Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1}E_c'(Z\kappa_c + U) \\ &- (Z\kappa_c + U)'Z(Z'Z)^{-1}(2E_c'E_c + Z'F_c + F_c'Z)(Z'Z)^{-1}Z'(Z\kappa_c + U) \\ &+ 2(Z\kappa_c + U)'Z(Z'Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1}Z'(Z\kappa_c + U). \end{aligned}$$

Finally, we rearrange the right side according to their order of convergence and obtain that

- $2\kappa_c'\{Z'E_c'(Z'Z)^{-1}E_c' + F_c'\}Z\kappa_c - 4\kappa_c'(Z'E_c + E_c'Z)(Z'Z)^{-1}E_c'Z\kappa_c + 2\kappa_c'(Z'E_c + E_c'Z)(Z'Z)^{-1}Z'E_c + E_c'Z)\kappa_c - \kappa_c'(2E_c'E_c + Z'F_c + F_c'Z)\,\kappa_c = 2\kappa_c'E_c'Z(Z'Z)^{-1}Z'E_c\kappa_c - 2\kappa_c'E_c'E_c\kappa_c = -2(\alpha_{c*} - \alpha_c)^2 A_c'M\ A_c$;

- $4\kappa_c'Z'E_c(Z'Z)^{-1}E_c'U - 4\kappa_c'(Z'E_c + E_c'Z)(Z'Z)^{-1}E_c'U - 4\kappa_c'Z'E_c(Z'Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1}Z'U + 2\kappa_c'F_c'U + 2\kappa_c'Z'F_c(Z'Z)^{-1}Z'U + 4\kappa_c'(Z'E_c + E_c'Z)(Z'\ Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1}Z'U - 2\kappa_c'(2E_c'E_c + Z'F_c + F_c'Z)(Z'Z)^{-1}Z'\ U = 2(\alpha_{c*} - \alpha_c)[B_c'MU - 2A_c'ME_c(Z'Z)^{-1}Z'U - 2A_c'Z(Z'Z)^{-1}E_c'MU]$; and

- $2[U'E_c(Z'Z)^{-1}E_c'U + U'F_c(Z'Z)^{-1}Z'U - 2U'E_c(Z'Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1}Z'\ U] + 2U'Z(Z'Z)^{-1}[(Z'E_c + E_c'Z)(Z'Z)^{-1}(Z'E_c + E_c'Z) - E_c'E_c - Z'F_c](Z'Z)^{-1}Z'\ U$.

We now apply Lemma A1 to each of these terms. First, Lemma A1(*ii* and *iii*) imply that $A_c'MA_c = A_c'A_c - AcZ(Z'Z)^{-1}Z'A_c = O_{\mathbb{P}}(n)$. Second, $B_c'MU = B_c'U - B_c'Z(Z'Z)^{-1}Z'U$, and Lemma A1 (*ii* and *iii*) implies that $B_c'MU = O_{\mathbb{P}}(n)$. Furthermore, Lemma A1(*iv*) implies that $B_c'MU = o_{\mathbb{P}}(n)$. Third, $A_c'ME_c = A_c'E_c - A_c'Z(Z'Z)^{-1}Z'E_c$. Assumption 2 and Lemma A1(*ii*, *iii*, and *iv*) imply that $A_c'ME_c(Z'Z)^{-1}Z'U = o_{\mathbb{P}}(n)$. Fourth, $E_c'MU = E_cU - E_cZ(Z'Z)^{-1}Z'U = o_{\mathbb{P}}(n)$ by Lemma A1(*i* and *iv*), so that $A_c'Z(Z'Z)^{-1}E_c'MU = o_{\mathbb{P}}(n)$. Therefore, $B_c'MU - 2A_c'ME_c(Z'Z)^{-1}Z'U - 2A_c'Z(Z'Z)^{-1}E_c'MU = o_{\mathbb{P}}(n)$. Finally, we combine all terms in Lemma A1 and obtain that

$$2[U'E_c(Z'Z)^{-1}E_c'U + U'F_c(Z'Z)^{-1}Z'U - 2U'E_c(Z'Z)^{-1}(Z'E_c + E_c'Z)(Z'Z)^{-1}Z'U] + 2U'Z(Z'Z)^{-1}[(Z'E_c + E_c'Z)(Z'Z)^{-1}(Z'E_c + E_c'Z) - E_c'E_c - Z'F_c](Z'Z)^{-1}Z'U = o_{\mathbb{P}}(n).$$

All of these facts imply that $L_n^{(2)}(c;\alpha_c) = -2(\alpha_{c*} - \alpha_c)^2 A_c'MA_c + o_{\mathbb{P}}(n)$. ■

Proof of Lemma 2: It is proved in the text. ■

Proof of Lemma 3: Given Lemma 2, the proof is almost identical to the proof of theorem 1 of BCP. ■

Proof of Theorem 1: In fact, the inequality just above Theorem 1 implies that $QLR_n = QLR_n^{(\beta=0)}$ under $\mathcal{H}_{0,m}$, and Lemma 3(*ii*) implies that $QLR_n^{(\beta=0)} \Rightarrow \sup_{\gamma\in\Gamma} \mathcal{Z}(\gamma)^2$. The desired result follows. ■

Proof of Theorem 2: Before proving the claim, we let $\underline{\gamma}$ and $\bar{\gamma}$ be the lower and upper limit of Γ such that $\Gamma_j := [\gamma_j, \gamma_{j+1}]$ such that $\gamma_0 := \underline{\gamma}$, $\gamma_{\bar{m}+1} := \bar{\gamma}$, and for $j = 1, 2, \ldots, \bar{m}$, $\gamma_j := j$.

We now prove the stated claim. First, $\lim_{n\to\infty} \mathbb{P}(\widehat{m}_n > m_*) = \lim_{n\to\infty} \boldsymbol{\alpha}_n = 0$ by virtue of the size decay condition (ii). Second, the asymptotic Lemma A2(*i*) implies that if $c_n = o(n)$, for any $j < m_*$,

$\lim_{n\to\infty} \mathbb{P}(QLR_n^{(c+1)} > c_n) = 1$. This implies that if $\boldsymbol{\alpha}_n$ is selected to yield $c_n = o(n)$, the desired result follows. We note the following six properties (*i* to *vi*): (*i*) $\sup_{\gamma\in\Gamma_j} \mathcal{Z}(\gamma)^2 = \sup_{\gamma\in\Gamma_j}\{\max[0,\mathcal{Z}(\gamma)]^2 + \max[0,\mathcal{Z}(\gamma)]^2\} \leq \sup_{\gamma\in\Gamma_j}\max[0,\mathcal{Z}(\gamma)]^2 + \sup_{\gamma\in\Gamma_j}\max[0,\mathcal{Z}(\gamma)]^2\}$, so that for any $u > 0$,

$$\mathbb{P}\left(\sup_{\gamma\in\Gamma_j}\mathcal{Z}(\gamma)^2 \geq u^2\right) \leq \mathbb{P}\left(\sup_{\gamma\in\Gamma_j}\max[0,\mathcal{Z}(\gamma)]^2 \geq \frac{u^2}{2}\right) + \mathbb{P}\left(\sup_{\gamma\in\Gamma_j}\min[0,\mathcal{Z}(\gamma)]^2 \geq \frac{u^2}{2}\right)$$
$$= \mathbb{P}\left(\sup_{\gamma\in\Gamma_j}\mathcal{Z}(\gamma) \geq \frac{u}{\sqrt{2}}\right) + \mathbb{P}\left(\inf_{\gamma\in\Gamma_j}\mathcal{Z}(\gamma) \leq -\frac{u}{\sqrt{2}}\right) = 2\mathbb{P}\left(\sup_{\gamma\in\Gamma_j}\mathcal{Z}(\gamma) \geq \frac{u}{\sqrt{2}}\right)$$

by the fact that $\mathbb{P}(\inf_{\gamma\in\Gamma_j}\mathcal{Z}(\gamma) \leq -u/\sqrt{2}) = \mathbb{P}(\sup_{\gamma\in\Gamma_j}\mathcal{Z}(\gamma) \geq u/\sqrt{2})$, where the last equality holds from the symmetry of Gaussian process distribution. Therefore, for any $u > 0$,

$$\mathbb{P}\left(\sup_{\gamma\in\Gamma}\mathcal{Z}(\gamma)^2 \geq u^2\right) \leq 2\sum_{j=1}^{\bar{m}+1}\mathbb{P}\left(\sup_{\gamma\in\Gamma_j}\mathcal{Z}(\gamma) \geq \frac{u}{\sqrt{2}}\right). \tag{4}$$

(*ii*) Given the conditions, if we let $\sigma_* := \sup_{\gamma\in\Gamma}\mathrm{var}[\mathcal{Z}(\gamma)]^{1/2}$, for any γ, $|\mathcal{Z}(\gamma)/\sigma_*| \leq |\mathcal{Z}(\gamma)/\sigma^0(\gamma)| = |\mathcal{Z}^0(\gamma)|$, so that for any $u > 0$,

$$\mathbb{P}\left(\sup_{\gamma\in\Gamma_j}\frac{\mathcal{Z}(\gamma)}{\sigma_*} \geq u\right) \leq \mathbb{P}\left(\sup_{\gamma\in\Gamma_j}\mathcal{Z}^0(\gamma) \geq u\right). \tag{5}$$

(*iii*) Lemma 7.1 of Piterbarg (1996) implies that as $u \to \infty$,

$$\mathbb{P}\left(\sup_{\gamma\in\Gamma_j}\mathcal{B}^S(\gamma) \geq u\right) = H_\delta\mu(\Gamma_j)u^{2/\delta}(1-\Phi(u))(1+o(1)), \tag{6}$$

where $\Phi(\cdot)$ is the distribution function of the standard normal random variable, $\mu(\cdot)$ is the Lebesgue measure of the given argument, $H_\delta := \lim_{\bar{\gamma}\to\infty} H(\bar{\gamma})/\bar{\gamma}$, and $H(\bar{\gamma}) := E[\exp(\max_{\gamma\in[0,\bar{\gamma}]}\mathcal{B}^F(\gamma))]$. Here, $\mathcal{B}^F(\cdot)$ is a fractional Brownian motion with mean $-|\gamma|^\delta$ and $\mathrm{cov}(\mathcal{B}^F(\gamma), \mathcal{B}^F(\gamma')) = |\gamma|^\delta + |\gamma'|^\delta - |\gamma-\gamma'|^\delta$ on Γ.

(*iv*) The Slepian inequality implies that for any v, $\mathbb{P}(\sup_\gamma \mathcal{Z}^0(\gamma) \geq v) \leq \mathbb{P}(\sup_\gamma \mathcal{B}^S(\gamma) \geq v)$ (*e.g.*, Piterbarg, 1996, p.6). Therefore, the Slepian inequality, (5), and (6) imply that as $u \to \infty$,

$$\mathbb{P}\left(\sup_{\gamma\in\Gamma_j}\mathcal{Z}(\gamma) \geq \frac{u}{\sqrt{2}}\right) \leq H_\delta\mu(\Gamma_j)\left(\frac{u}{\sqrt{2}\sigma_*}\right)^{2/\delta}\left(1-\Phi\left(\frac{u}{\sqrt{2}\sigma_*}\right)\right)(1+o(1)), \tag{7}$$

so that it follows that

$$\mathbb{P}\left(\sup_{\gamma\in\Gamma}\mathcal{Z}(\gamma)^2 \geq u^2\right) \leq 2H_\delta\mu_*\left(\frac{u}{\sqrt{2}\sigma_*}\right)^{2/\delta}\left(1-\Phi\left(\frac{u}{\sqrt{2}\sigma_*}\right)\right)(1+o(1))$$

by (4), where $\mu_* := \mu(\Gamma)$.

(*v*) We further note that $1-\Phi(\cdot) = \frac{1}{2}\mathrm{erfc}((\cdot)/\sqrt{2}) \leq \frac{1}{2}\exp(-(\cdot)^2/2)$. Hence, if $u\to\infty$, it follows from

$$\mathbb{P}\left(\sup_{\gamma\in\Gamma}\mathcal{Z}(\gamma)^2 \geq u^2\right) \leq H_\delta\mu_*\left(\frac{u^2}{2\sigma_*^2}\right)^{1/\delta}\exp\left(-\frac{u^2}{4\sigma_*^2}\right)(1+o(1)). \tag{8}$$

(*vi*) Finally, if we let the left side of (8) and u^2 be the significance level $\boldsymbol{\alpha}_n$ and its associated critical value c_n, respectively, then

$$-\frac{\log(\boldsymbol{\alpha}_n)}{n} \geq -\frac{1}{\delta}\frac{\log(c_n)}{n} + \frac{1}{4\sigma_*^2}\frac{c_n}{n} + o(1)$$

by noting that $\left\{\log\left(H_\delta\mu_*\right) - \frac{1}{\delta}\log(2\sigma_*^2)\right\} = O(1)$. We now note that

$$-\frac{1}{\delta}\frac{\log(c_n)}{n} + \frac{1}{4\sigma_*^2}\frac{c_n}{n} = \frac{1}{4\sigma_*^2}\frac{c_n}{n}\left(1-\frac{4\sigma_*^2}{\delta}\frac{\log(c_n)}{c_n}\right) = \frac{1}{4\sigma_*^2}\frac{c_n}{n}(1+o(1))$$

as $c_n\to\infty$. Therefore, if $\log(\boldsymbol{\alpha}_n) = o(n)$, as is assumed in condition (iii), it follows that $c_n = o(n)$. This completes the proof. ■

Proof of Theorem 3: Weak convergence of the quasi-likelihood ratio (QLR) test statistic is proved in the same way as that of Theorem 1, so we only derive the covariance kernel of $\widetilde{\mathcal{Z}}(\cdot)$.

First, note that applying Theorem 1 implies that $QLR_n = \sup_{\gamma\in\Gamma}\left\{S(\gamma)'MU\right\}^2/\{\widehat{\sigma}_{n,0}^2 S(\gamma)'MS(\gamma)\}$ under $\widetilde{\mathcal{H}}_0$. Next, applying the uniform law of large numbers (ULLN) to $n^{-1}S(\cdot)'MS(\cdot)$ shows that $\sup_{\gamma\in\Gamma}$ $|n^{-1}\widehat{\sigma}_{n,0}^2 S(\gamma)'MS(\gamma) - \widetilde{\sigma}^2(\gamma,\gamma)| \overset{\text{a.s.}}{\to} 0$, where for each γ,

$$\widetilde{\sigma}^2(\gamma,\gamma) := \sigma_*^2\{\widetilde{A}_{4,4}(\gamma) - \widetilde{A}^{3,1}(\gamma)(\widetilde{A}^{1,1})^{-1}\widetilde{A}^{1,3}(\gamma)\} = \frac{\sigma_*^2\prod_{i=0}^m(\gamma-i)^2}{(2\gamma+1)\prod_{i=0}^m(\gamma+i+1)^2}.$$

Also note that for each γ,

$$\frac{1}{\sqrt{n}}(S(\gamma)'MU) = \frac{1}{\sqrt{n}}\sum u_t s_{n,t}^\gamma - \widetilde{A}^{3,1}(\gamma)(\widetilde{A}^{1,1})^{-1}\frac{1}{\sqrt{n}}\sum u_t z_{n,t} + o_{\mathbb{P}}(1),$$

so that, if we let $\widetilde{\mathcal{G}}(\cdot)$ be the weak limit of $n^{-1/2}S(\gamma)'MU$, we have

$$\begin{aligned}\mathbb{E}[\widetilde{\mathcal{G}}(\gamma)\widetilde{\mathcal{G}}(\gamma')] &= \widetilde{B}_{4,4}(\gamma,\gamma') - \widetilde{A}^{3,1}(\gamma)(\widetilde{A}^{1,1})^{-1}\widetilde{B}^{1,3}(\gamma') \\ &\qquad - \widetilde{A}^{3,1}(\gamma')(\widetilde{A}^{1,1})^{-1}\widetilde{B}^{1,3}(\gamma) + \widetilde{A}^{3,1}(\gamma)(\widetilde{A}^{1,1})^{-1}\widetilde{B}^{1,1}(\widetilde{A}^{1,1})^{-1}\widetilde{A}^{1,3}(\gamma') \\ &= \frac{\sigma_*^2\prod_{i=0}^m(\gamma-i)(\gamma'-i)}{(\gamma+\gamma'+1)\prod_{i=0}^m(\gamma+i+1)(\gamma'+i+1)}.\end{aligned}$$

This implies that

$$\mathbb{E}[\widetilde{\mathcal{Z}}(\gamma)\widetilde{\mathcal{Z}}(\gamma')] = \frac{\{\prod_{i=0}^m(\gamma-i)(\gamma'-1)\}(2\gamma+1)^{1/2}(2\gamma'+1)^{1/2}}{\{\prod_{i=0}^m|\gamma-i|\cdot|\gamma'-i|\}(\gamma+\gamma'+1)} = c_m(\gamma,\gamma')\frac{(2\gamma+1)^{1/2}(2\gamma'+1)^{1/2}}{(\gamma+\gamma'+1)}$$

by the definition of $c_m(\gamma,\gamma') := \prod_{i=0}^m(\gamma-i)(\gamma'-i)/|\prod_{i=0}^m(\gamma-i)(\gamma'-i)|$, as desired. ■

4 Additional Useful Properties

4.1 Theoretical Part

In this subsection, we provide some additional properties that are not contained in the paper but useful in obtaining the null limit distribution of the QLR test statistic in Section 3 of Cho and Phillips (2017).

First, for each γ, the almost sure limit of $n^{-1}\sum\widetilde{G}_t(\gamma)\widetilde{G}_t(\gamma)'$ that is denoted as $\widetilde{A}(\gamma)$ can be provided as follows:

$$\widetilde{A}(\gamma) := \left[\begin{array}{c|cc}\widetilde{A}^{1,1} & \widetilde{A}^{1,2} & \widetilde{A}^{1,3}(\gamma) \\ \hline \widetilde{A}^{2,1} & \widetilde{A}_{3,3} & \widetilde{A}_{3,4}(\gamma) \\ \widetilde{A}^{3,1}(\gamma) & \widetilde{A}_{4,3}(\gamma) & \widetilde{A}_{4,4}(\gamma)\end{array}\right] := \left[\begin{array}{cc|cc}\widetilde{A}_{1,1} & \widetilde{A}_{1,2} & \widetilde{A}_{1,3} & \widetilde{A}_{1,4}(\gamma) \\ \widetilde{A}_{2,1} & \widetilde{A}_{2,2} & \widetilde{A}_{2,3} & \widetilde{A}_{2,4}(\gamma) \\ \hline \widetilde{A}_{3,1} & \widetilde{A}_{3,2} & \widetilde{A}_{3,3} & \widetilde{A}_{3,4}(\gamma) \\ \widetilde{A}_{4,1}(\gamma) & \widetilde{A}_{4,2}(\gamma) & \widetilde{A}_{4,3}(\gamma) & \widetilde{A}_{4,4}(\gamma)\end{array}\right],$$

where $\widetilde{G}_t(\gamma) := [\, s_t(m)', d_t',\ s_t(m)'\log(s_{n,t}), s_{n,t}^{\gamma}]'$, and the submatrices are defined as follows: for $i, j = 1, 2, \ldots, m+1$,

$$\underset{(m+1)\times(m+1)}{\widetilde{A}_{1,1}} := \left[\frac{1}{i+j-1}\right], \quad \underset{(m+1)\times k}{\widetilde{A}_{1,2}} := \left[\frac{\mathbb{E}[d_t']}{j}\right], \quad \underset{(m+1)\times(m+1)}{\widetilde{A}_{1,3}} := \left[\frac{-1}{(i+j-1)^2}\right],$$

$$\underset{(m+1)\times 1}{\widetilde{A}_{1,4}(\gamma)} := \left[\frac{1}{\gamma+j}\right], \quad \underset{k\times k}{\widetilde{A}_{2,2}} := \mathbb{E}[d_t d_t'], \quad \underset{k\times(m+1)}{\widetilde{A}_{2,3}} := \left[-\frac{\mathbb{E}[d_t]}{j^2}\right], \quad \underset{k\times 1}{\widetilde{A}_{2,4}(\gamma)} := \left[\frac{\mathbb{E}[d_t]}{\gamma+1}\right],$$

$$\underset{(m+1)\times(m+1)}{\widetilde{A}_{3,3}} := \left[\frac{2}{(i+j-1)^3}\right], \quad \underset{(m+1)\times 1}{\widetilde{A}_{3,4}(\gamma)} := \left[\frac{-1}{(\gamma+j)^2}\right], \quad \text{and} \quad \underset{1\times 1}{\widetilde{A}_{4,4}(\gamma)} := \frac{1}{2\gamma+1}.$$

Since $\widetilde{A}(\gamma)$ is supposed to be symmetric, we let $\widetilde{A}_{2,1} := \widetilde{A}'_{1,2}$, $\widetilde{A}_{3,1} := \widetilde{A}'_{1,3}$, $\widetilde{A}_{4,1}(\gamma) := \widetilde{A}_{1,4}(\gamma)'$, $\widetilde{A}_{2,3} := \widetilde{A}'_{3,2}$, $\widetilde{A}_{2,4}(\gamma) := \widetilde{A}_{4,2}(\gamma)'$, and $\widetilde{A}_{4,3} := \widetilde{A}'_{3,4}$. Observe that $\widetilde{A}(\gamma)$ corresponds to $A(\gamma)$ in Section 2 of Cho and Phillips (2017). Note that some of the elements are obtained by using the fact that $s_t(m)$ is not a random covariate.

Second, for each γ and γ', the almost sure limit of $n^{-1}\sum u_t^2 \widetilde{G}_t(\gamma)\widetilde{G}_t(\gamma')'$ that is denoted as $\widetilde{B}(\gamma,\gamma')$ is provided as follows:

$$\widetilde{B}(\gamma,\gamma') := \left[\begin{array}{c|cc} \widetilde{B}^{1,1} & \widetilde{B}^{1,2} & \widetilde{B}^{1,3}(\gamma') \\ \hline \widetilde{B}^{2,1} & \widetilde{B}_{3,3} & \widetilde{B}_{3,4}(\gamma') \\ \widetilde{B}^{3,1}(\gamma) & \widetilde{B}_{4,3}(\gamma) & \widetilde{B}_{4,4}(\gamma,\gamma') \end{array}\right] := \left[\begin{array}{cc|cc} \widetilde{B}_{1,1} & \widetilde{B}_{1,2} & \widetilde{B}_{1,3} & \widetilde{B}_{1,4}(\gamma') \\ \widetilde{B}_{2,1} & \widetilde{B}_{2,2} & \widetilde{B}_{2,3} & \widetilde{B}_{2,4}(\gamma') \\ \hline \widetilde{B}_{3,1} & \widetilde{B}_{3,2} & \widetilde{B}_{3,3} & \widetilde{B}_{3,4}(\gamma') \\ \widetilde{B}_{4,1}(\gamma) & \widetilde{B}_{4,2}(\gamma) & \widetilde{B}_{4,3}(\gamma) & \widetilde{B}_{4,4}(\gamma,\gamma') \end{array}\right],$$

where the submatrices are defined below, for $i, j = 1, 2, \ldots, m+1$,

$$\underset{(m+1)\times(m+1)}{\widetilde{B}_{1,1}} := \left[\frac{\mathbb{E}[u_t^2]}{i+j-1}\right], \quad \underset{(m+1)\times k}{\widetilde{B}_{1,2}} := \left[\frac{\mathbb{E}[u_t^2 d_t']}{j}\right], \quad \underset{(m+1)\times(m+1)}{\widetilde{B}_{1,3}} := \left[\frac{-\mathbb{E}[u_t^2]}{(i+j-1)^2}\right],$$

$$\underset{(m+1)\times 1}{\widetilde{B}_{1,4}(\gamma')} := \left[\frac{\mathbb{E}[u_t^2]}{\gamma'+j}\right], \quad \underset{k\times k}{\widetilde{B}_{2,2}} := \mathbb{E}[u_t^2 d_t d_t'], \quad \underset{k\times(m+1)}{\widetilde{B}_{2,3}} := \left[-\frac{\mathbb{E}[u_t^2 d_t]}{j^2}\right], \quad \underset{k\times 1}{\widetilde{B}_{2,4}(\gamma')} := \left[\frac{\mathbb{E}[u_t^2 d_t]}{\gamma'+1}\right],$$

$$\underset{(m+1)\times(m+1)}{\widetilde{B}_{3,3}} := \left[\frac{2\mathbb{E}[u_t^2]}{(i+j-1)^3}\right], \quad \underset{(m+1)\times 1}{\widetilde{B}_{3,4}(\gamma')} := \left[\frac{-\mathbb{E}[u_t^2]}{(\gamma'+j)^2}\right], \quad \underset{1\times 1}{\widetilde{B}_{4,4}(\gamma,\gamma')} := \frac{\mathbb{E}[u_t^2]}{\gamma+\gamma'+1},$$

where $u_t := y_t - \mathbb{E}[y_t|d_t]$. As $\widetilde{B}(\gamma,\gamma)$ is symmetric, $\widetilde{B}_{2,1} := \widetilde{B}'_{1,2}$, $\widetilde{B}_{3,1} := \widetilde{B}'_{1,3}$, $\widetilde{B}_{4,1}(\gamma) := \widetilde{B}_{1,4}(\gamma)'$, $\widetilde{B}_{2,3} := \widetilde{B}'_{3,2}$, $\widetilde{B}_{2,4}(\gamma) := \widetilde{B}_{4,2}(\gamma)'$, and $\widetilde{B}_{4,3} := \widetilde{B}'_{3,4}$. The matrix $\widetilde{B}(\gamma,\gamma)$ corresponds to $B(\gamma)$ in Section 2 of Cho and Phillips (2017).

Third, we show that $\widetilde{A}(\cdot)$ is positive definite if and only if the covariance matrix of d_t is positive definite. We reorganize $\widetilde{A}(\cdot)$ into

$$\left[\begin{array}{c|c} \mathbb{A}^{1,1}(\gamma) & \mathbb{A}^{1,2}(\gamma) \\ \hline \mathbb{A}^{2,1}(\gamma) & \widetilde{A}_{2,2} \end{array}\right] := \left[\begin{array}{ccc|c} & & & \widetilde{A}_{1,2} \\ & \mathbb{A}^{1,1}(\gamma) & & \widetilde{A}_{3,2} \\ & & & \widetilde{A}_{4,2}(\gamma) \\ \hline \widetilde{A}_{2,1} & \widetilde{A}_{2,3} & \widetilde{A}_{2,4}(\gamma) & \widetilde{A}_{2,2} \end{array}\right],$$

where

$$\mathbb{A}^{1,1}(\gamma) := \left[\begin{array}{c|c} \mathbb{A}_{1,1} & \mathbb{A}_{1,2}(\gamma) \\ \hline \mathbb{A}_{2,1}(\gamma) & \widetilde{A}_{4,4}(\gamma) \end{array}\right] := \left[\begin{array}{cc|c} \widetilde{A}_{1,1} & \widetilde{A}_{1,3} & \widetilde{A}_{1,4}(\gamma) \\ \widetilde{A}_{3,1} & \widetilde{A}_{3,3} & \widetilde{A}_{3,4}(\gamma) \\ \hline \widetilde{A}_{4,1}(\gamma) & \widetilde{A}_{4,3}(\gamma) & \widetilde{A}_{4,4}(\gamma) \end{array}\right],$$

$\mathbb{A}_{1,1}$ is positive definite by the definitions of $\widetilde{A}_{1,1}$, $\widetilde{A}_{1,3}$, and $\widetilde{A}_{3,3}$. This in turn implies that $\mathbb{A}^{1,1}(\gamma)$ is positive definite uniformly for each $\gamma \in \Gamma(\epsilon)$ if and only if $\widetilde{A}_{4,4}(\gamma) - \mathbb{A}_{2,1}(\gamma)\mathbb{A}_{1,1}^{-1}\mathbb{A}_{1,2}(\gamma)$ is positive definite. Some algebra shows that

$$\widetilde{A}_{4,4}(\gamma) - \mathbb{A}_{2,1}(\gamma)\mathbb{A}_{1,1}^{-1}\mathbb{A}_{1,2}(\gamma) = \frac{\prod_{i=0}^{m}(\gamma - i)^4}{(1+2\gamma)\prod_{i=0}^{m}(\gamma + i + 1)^4},$$

which is strictly greater than zero for each $\gamma \in \Gamma(\epsilon)$, implying that $\mathbb{A}^{1,1}(\cdot)$ is positive definite uniformly on $\Gamma(\epsilon)$. This further implies that $\widetilde{A}(\cdot)$ is positive definite uniformly on $\Gamma(\epsilon)$ if and only if for each $\gamma \in \Gamma(\epsilon)$, $\widetilde{A}_{2,2} - \mathbb{A}^{2,1}(\gamma)\mathbb{A}^{1,1}(\gamma)^{-1}\mathbb{A}^{1,2}(\gamma)$ is positive definite. Here, every column of $\mathbb{A}^{1,2}(\gamma)$ is a linear transformation of the first column of $\mathbb{A}^{1,1}(\gamma)$, so that $\widetilde{A}_{2,2} - \mathbb{A}^{2,1}(\gamma)\mathbb{A}^{1,1}(\gamma)^{-1}\mathbb{A}^{1,2}(\gamma) = \mathbb{E}[d_t d_t'] - \mathbb{E}[d_t]\mathbb{E}[d_t]'$ that is the covariance matrix of d_t. Therefore, $\widetilde{A}(\cdot)$ is positive definite uniformly on $\Gamma(\epsilon)$ if and only if the covariance matrix of d_t is positive definite, that is provided the elements of d_t are not linearly dependent almost surely.

4.2 Simulation Part

We tabulate asymptotic critical values obtained by simulating $\sup_{\gamma\in\Gamma} \bar{\mathcal{Z}}_\ell(\gamma)^2$ for large ℓ and various assumptions on Γ, where $\bar{\mathcal{Z}}_\ell(\cdot)$ is the truncated exponential Gaussian process. The critical values of BCP should be used only when $m = 1$. Table 1 reports critical values for the QLR test for models with polynomials of degree $m =$ 2, 3, 4, 5, 6, 7, 8, 9, 10. With these tabulated results, users can test for neglected nonlinearity up to a 10-th degree polynomial null model. The values reported are obtained with $\ell = 1000$ and one million replications. Since this methodology provides more precise critical values than those in BCP, we include the $m = 1$ case in Table 1. Interested readers can also download the GAUSS program code that generates the null limit distribution from the following URL:

`http://web.yonsei.ac.kr/jinseocho/polynomial.htm`.

Users can select different values of the lower and upper bounds of Γ, ℓ, and the number of replications in running the code.

4.3 Empirical Part

In addition to the empirical results in Cho and Phillips (2017), we also report additional empirical findings from Bierens and Ginther's (2001) data set. Their original samples are drawn from males aged between 18 and 70 with annual income greater than USD 50 in 1992. For our analysis, we focus on full-time workers, and the sample size is 25,631 by this restriction. From this analysis, we intend to ignore different cohort effects and introduce more heterogeneity to the data. For more information on the original data set, readers can refer to Bierens and Ginther (2001).

The test results are contained in Table 2. The table structure is identical to that in Table 3 in Cho and Phillips (2017). We summarize the findings as follows.

First, our empirical analysis shows that the most parsimonious polynomial orders for m_1 and m_2 are 4 and 6 for every model and data set. If the polynomial degree of schooling years or experience is less than or equal to 2 or 4, respectively, every null model is rejected. This finding shows that the Mincer equation does not hold for the 1988 CPS data.

Second, a further higher-degree polynomial model is required than what Murphy and Welch (1990) and Lemieux (2006) found from their data. Our estimated mode is quadratic with respect to the schooling years, which is consistent with Lemieux's (2006) observation, but our model is hexic with respect to experience. This result is consistent with what Bierens and Ginther (2001) implicitly obtained from their LAD estimation.

Third, the quartic model is not flexible enough to detect the nonlinearity of the earnings equation for this data set, and this weakness arises mainly from the tail levels of experience. Figure 1 is drawn in the same way as Figure 1 in Cho and Phillips (2017). Evidently, the quartic model underestimates log earnings in the right-tail, and this deficiency is remedied by the hexic function.

Finally, these estimations also show that the respective degrees of polynomial nonlinearity with respect to schooling years and experience in the original Mincer equation are variant to data and/or inclusion of other explanatory variables in the model, thereby indicating the need for some flexibility in treating potential nonlinearity in these key variables, as is possible with flexible polynomial specifications and, more generally, with sieve approximants.

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Levels \ Γ	$[-0.20, 1.50]$	$[-0.10, 1.50]$	$[0.00, 1.50]$	$[0.10, 1.50]$
10%	3.7336	3.5869	3.4772	3.4003
5%	5.0114	4.8423	4.7283	4.6434
1%	8.0323	7.8151	7.7430	7.6375
Levels \ Γ	$[-0.20, 2.50]$	$[-0.10, 2.50]$	$[0.00, 2.50]$	$[0.10, 2.50]$
10%	3.8966	3.7750	3.6651	3.5822
5%	5.1831	5.0589	4.9339	4.8459
1%	8.2617	8.1332	7.9663	7.8625
Levels \ Γ	$[-0.20, 3.50]$	$[-0.10, 3.50]$	$[0.00, 3.50]$	$[0.10, 3.50]$
10%	4.0125	3.8996	3.8050	3.7358
5%	5.3049	5.1925	5.0956	5.0150
1%	8.3942	8.2808	8.1330	8.0578
Levels \ Γ	$[-0.20, 4.50]$	$[-0.10, 4.50]$	$[0.00, 4.50]$	$[0.10, 4.50]$
10%	4.0975	3.9859	3.8874	3.8128
5%	5.4021	5.2884	5.1750	5.0841
1%	8.5032	8.3619	8.2586	8.1464
Levels \ Γ	$[-0.20, 5.50]$	$[-0.10, 5.50]$	$[0.00, 5.50]$	$[0.10, 5.50]$
10%	4.1702	4.0576	3.9581	3.8978
5%	5.4927	5.3664	5.2487	5.1970
1%	8.5837	8.4411	8.3105	8.2641
Levels \ Γ	$[-0.20, 6.50]$	$[-0.10, 6.50]$	$[0.00, 6.50]$	$[0.10, 6.50]$
10%	4.2150	4.1058	4.0209	3.9663
5%	5.5267	5.4220	5.3256	5.2666
1%	8.6134	8.5069	8.4181	8.3524
Levels \ Γ	$[-0.20, 7.50]$	$[-0.10, 7.50]$	$[0.00, 7.50]$	$[0.10, 7.50]$
10%	4.2587	4.1599	4.0652	4.0051
5%	5.5725	5.4723	5.3720	5.2999
1%	8.6938	8.5761	8.4599	8.3650
Levels \ Γ	$[-0.20, 8.50]$	$[-0.10, 8.50]$	$[0.00, 8.50]$	$[0.10, 8.50]$
10%	4.3033	4.1951	4.1135	4.0538
5%	5.6144	5.5156	5.4253	5.3551
1%	8.7141	8.6312	8.4897	8.4218
Levels \ Γ	$[-0.20, 9.50]$	$[-0.10, 9.50]$	$[0.00, 9.50]$	$[0.10, 9.50]$
10%	4.3351	4.2366	4.1557	4.0880
5%	5.6507	5.5505	5.4726	5.3905
1%	8.7754	8.6351	8.5425	8.4747
Levels \ Γ	$[-0.20, 10.50]$	$[-0.10, 10.50]$	$[0.00, 10.50]$	$[0.10, 10.50]$
10%	4.3652	4.2769	4.1752	4.1244
5%	5.6828	5.5892	5.4841	5.4492
1%	8.8038	8.7053	8.5877	8.5292

Table 1: ASYMPTOTIC CRITICAL VALUES OF THE QLR TEST STATISTIC. This table contains the asymptotic critical values obtained by generating the truncated exponential Gaussian process 1,000,000 times.

Null Model 1: $\alpha_{0*} + \sum_{j=1}^{m_1} \beta_{j*} s_t^j + \sum_{j=1}^{m_2} \alpha_{j*} x_t^j$									
$m_2 \setminus m_1$	1	2	4	6	$m_2 \setminus m_1$	1	2	4	6
1	**2094.8**	**2135.2**	**2153.7**	**2146.0**	1	**68.36**	**26.28**	2.76	0.11
	(0.00)	**(0.00)**	**(0.00)**	**(0.00)**		**(0.00)**	**(0.00)**	(17.00)	(87.60)
2	**404.50**	**372.11**	**380.44**	**372.28**	2	**147.70**	**38.84**	0.72	0.12
	(0.00)	**(0.00)**	**(0.00)**	**(0.00)**		**(0.00)**	**(0.00)**	(58.40)	(87.60)
4	**62.34**	**51.71**	**42.67**	**52.04**	4	**102.74**	**47.84**	1.08	0.04
	(0.00)	**(0.00)**	**(0.00)**	**(0.00)**		**(0.00)**	**(0.00)**	(43.00)	(95.40)
6	0.98	2.30	*2.58*	2.88	6	**107.97**	**45.59**	*1.06*	0.00
	(53.80)	(28.00)	(*28.20*)	(19.80)		**(0.00)**	**(0.00)**	(*44.60*)	(99.20)

Null Model 2: $\alpha_{0*} + \sum_{j=1}^{m_1} \beta_{j*} s_t^j + \sum_{j=1}^{m_2} \alpha_{j*} x_t^j + \eta_{1*} b_t + sm_t' \eta_{2*}$									
$m_2 \setminus m_1$	1	2	4	6	$m_2 \setminus m_1$	1	2	4	6
1	**2244.0**	**2278.5**	**2292.5**	**2283.0**	1	**45.29**	**17.51**	4.53	0.10
	(0.00)	**(0.00)**	**(0.00)**	**(0.00)**		**(0.00)**	**(0.00)**	(5.40)	(10.78)
2	**429.14**	**399.64**	**400.48**	**396.35**	2	**110.66**	**28.37**	1.05	0.11
	(0.00)	**(0.00)**	**(0.00)**	**(0.00)**		**(0.00)**	**(0.00)**	(43.40)	(89.40)
4	**67.69**	**56.28**	**48.73**	**58.33**	4	**82.47**	**35.79**	1.44	0.02
	(0.00)	**(0.00)**	**(0.00)**	**(0.00)**		**(0.00)**	**(0.00)**	(37.60)	(96.80)
6	3.20	3.21	*3.34*	3.70	6	**76.30**	**33.51**	*0.59*	0.00
	(10.40)	(13.20)	(*12.40*)	(10.20)		**(0.00)**	**(0.00)**	(*66.00*)	(100.0)

Table 2: INFERENCES FOR THE MINCER EQUATION USING ALL OBSERVATIONS. This table shows the QLR test statistic and its p-values that are obtained by the data set in Bierens and Ginther (2001). The sample size is 25,631. Boldface p-values indicate significance levels less than 0.1%. Refer to Table 3 in Cho and Phillips (2017) for more information.

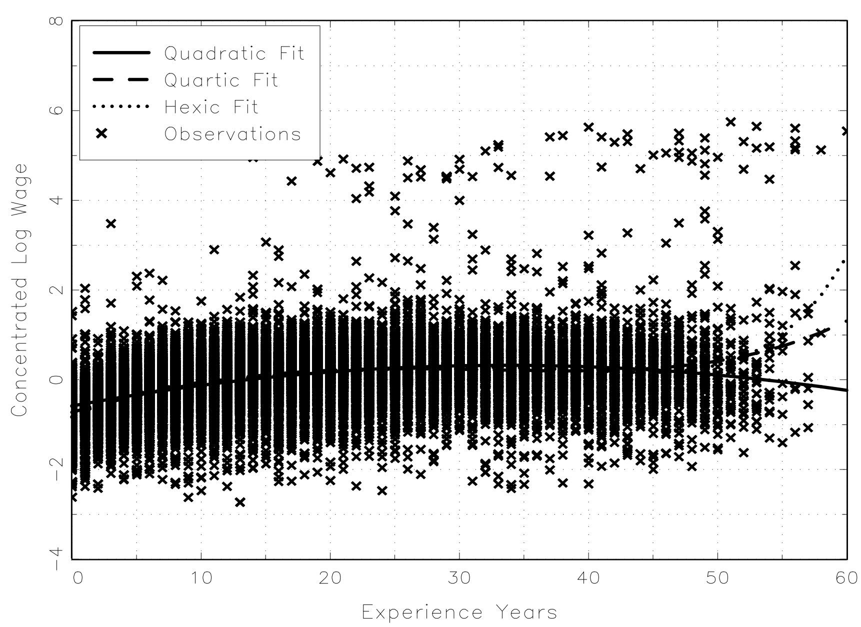

Figure 1: QUADRATIC, QUARTIC, HEXIC FITS OF CONCENTRATED LOG WAGES AGAINST EXPERIENCE. This figure shows the quadratic, quartic, and hexic fits of concentrated log wages with respect to experience. The concentrated log wages are the prediction errors obtained by regressing log wages against schooling years and the other dummy variables in (7) in Cho and Phillips (2017): b_t and sm_t.