Mathematical Proofs for "Testing for Unobserved Heterogeneity in Exponential and Weibull Duration Models"

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Abstract

We provide mathematical proofs for the results in "Testing for Unobserved Heterogeneity in Exponential and Weibull Duration Models" by Cho and White (2010).

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1 Introduction

This note provides mathematical proofs of the results stated Cho and White (2010). We indicate the equation numbers in Cho and White (2010) using square brackets.

2 Proofs

Proof of Theorem 1: For an efficient presentation, we first prove the result in (iv); we then prove the remaining results in reverse order as corollaries. We separately derive the covariance structures given in [3] and [4].

(iv) To show weak convergence, we verify the conditions of theorem 6(a) of Cho and White (2007). Our assumptions A1 to A3 and $\mathcal{H}_o$ are sufficient for their assumptions A1, A2(i, iii), A3, A4, and A5(ii, iii); further, under $\mathcal{H}_o$, our assumption A4 relaxes their A6(iv).[1] The LR statistic thus satisfies the sufficient conditions for the stated weak convergence, together with tightness.

Next, we derive the covariance structure [4]. For this, we use the formula given in lemma 1(b) of Cho and White (2007a). For each α and α',

$$E[r_t(\alpha)r_t(\alpha')] = \frac{\alpha\alpha'}{\alpha+\alpha'-1}, \tag{1}$$

$$E[r_t(\alpha)s_t] = \left(\frac{\alpha-1}{\alpha}\right)[\,1,\ \ E[d_t^*]',\ \ -\ln(\alpha)/\gamma^* + E(\phi_t^*)\,]', \quad \text{and} \tag{2}$$

$$E[s_t s_t'] = \begin{bmatrix} 1 & E[d_t^{*\prime}] & E[\phi_t^*] \\ E[d_t^*] & E[d_t^* d_t^{*\prime}] & E[d_t^*\phi_t^*] \\ E[\phi_t^*] & E[\phi_t^* d_t^{*\prime}] & E[\phi_t^{*2}] + \pi^2/(6\gamma^{*2}) \end{bmatrix}, \tag{3}$$

where $d_t^* \equiv \nabla_\beta \ln[g(X_t;\beta^*)]$ and

$$\phi_t^* \equiv E\left\{\frac{\partial}{\partial\delta}\ln[f(Y_t \mid X_t;\delta^*,\beta^*,\gamma^*)]\frac{\partial}{\partial\gamma}\ln[f(Y_t \mid X_t;\delta^*,\beta^*,\gamma^*)] \mid X_t\right\}.$$

[1] To have a well-defined Gaussian limit for every $\alpha(\neq 1)$, it is not necessary to impose the positive definite matrix assumption on $\mathbf{A}(\alpha,\alpha')$ for every $(\alpha,\alpha')(\neq(1,1))$ as in Cho and White (2007, assumption A6(iv)). Our assumption A4 relaxes this restriction.

Also, $E[r_t(\alpha)r_t(\alpha')]$ cannot be computed without having $\alpha + \alpha' - 1 > 0$, which is ensured by our assumption that $\inf A > 1/2$. Plugging these into

$$\rho_2(\alpha, \alpha') \equiv E[r_t(\alpha)r_t(\alpha')] - 1 - E[r_t(\alpha)s_t]'\{E[s_t s_t']\}^{-1}E[r_t(\alpha')s_t] \quad (4)$$

yields

$$\rho_2(\alpha, \alpha') = \left(\frac{\alpha - 1}{\alpha}\right)\left(\frac{\alpha' - 1}{\alpha'}\right)\left[\frac{(\alpha-1)(\alpha'-1)}{(\alpha+\alpha'-1)} - \frac{\xi^*}{\gamma^{*2}}\ln(\alpha)\ln(\alpha')\right], \quad (5)$$

which implies the standardized covariance structure: $\rho_2(\alpha, \alpha')/\{\rho_2(\alpha, \alpha)^{1/2}\rho_2(\alpha', \alpha')^{1/2}\}$ identical to [4], so the proof is complete.

(iii) If $\beta^* = 0$, then for each α and α', $E[r_t(\alpha)r_t(\alpha')]$ is identical to (1); $E[r_t(\alpha)s_t]$ is equal to $E[r_t(\alpha)s_t]$ in (2) without $E[d_t^*]'$; finally $E[s_t s_t']$ is now identical to $E[s_t s_t']$ in (3) without the elements in the second column and second row blocks. Given that $\beta^* = 0$, ϕ_t^* is a constant, implying that $\text{var}[\phi_t^*] = 0$ and $\text{cov}[\phi_t^*, d_t^*] = \mathbf{0}$. Therefore, $\xi^* = 6\gamma^{*2}/\pi^2$. Using these, it is straightforward to derive the covariance structure given in the statement of the Theorem.

(ii) If β^* is unknown and $\gamma^* = 1$, then for each α and α', $E[r_t(\alpha)r_t(\alpha')]$ is identical to (1); $E[r_t(\alpha)s_t]$ is equal to $E[r_t(\alpha)s_t]$ in (2) without the last column element; finally $E[s_t s_t']$ is now identical to $E[s_t s_t']$ in (3) without the elements in the third column and third row blocks. Plugging these into the covariance function in (4) and (5), it follows that the standardized covariance structure is identical to [3].

(i) Finally, if $\beta^* = 0$, then the covariance structure does not involve d_t^*, so that $E[r_t(\alpha)s_t] = (\alpha - 1)/\alpha$, $E[s_t s_t'] = 1$, and $E[r_t(\alpha)r_t(\alpha')]$ is the same as before. Using these, we obtain the same covariance structure as in (ii). ■

Proof of Corollary 1: Under Cox's (1972) proportional hazard assumption, $d_t^* = X_t$ and $\phi_t^* = -\{\ln(\zeta^*) + \tilde{\gamma} + X_t'\beta^*\}/\gamma^*$, so that $\text{var}[\phi_t^*] = \beta^{*\prime}\text{var}[X_t]\beta^*/\gamma^{*2}$, and $\text{cov}[\phi_t^*, d_t^*] = \text{var}[X_t]\beta^*/\gamma^*$, and $\text{var}[d_t^*] = \text{var}[X_t]$. This implies that $\text{var}[\phi_t^*] - \text{cov}[\phi_t^*, d_t^*]'\text{var}[d_t^*]^{-1}\text{cov}[\phi_t^*, d_t^*] = 0$. The desired result follows. ■

Proof of Theorem 2: (i) To show the result, we verify that for each α, $\alpha' \in A$, $E[\mathcal{G}_1(\alpha)\mathcal{G}_1(\alpha')] = E[\bar{\mathcal{G}}_1(\alpha)\bar{\mathcal{G}}_1(\alpha')]$. For this, note that for each α, α', $\rho_1(\alpha, \alpha')$ can be written

$$\rho_1(\alpha, \alpha') = \frac{(1-\alpha)(1-\alpha')/(\alpha\alpha')}{1 - (1-\alpha)(1-\alpha')/(\alpha\alpha')} - \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1-\alpha'}{\alpha'}\right).$$

For notational simplicity, we let q and q' be $(\alpha - 1)/\alpha$ and $(\alpha' - 1)/\alpha'$ respectively. Then

$$\rho_1(\alpha, \alpha') = qq'/(1 - qq') - qq' = \sum_{k=1}^{\infty}(qq')^k - qq' = \sum_{k=2}^{\infty}(qq')^k,$$

implying that

$$E[\mathcal{G}_1(\alpha)\mathcal{G}_1(\alpha')] = \sum_{k=2}^{\infty} \frac{(qq')^k}{(q^4/[1 - q^2])^{1/2}(q'^4/[1 - q'^2])^{1/2}}.$$

Note that $(q^4/[1 - q^2])^{-1/2}q^k = a_k(\alpha)$. Thus, $E[\mathcal{G}_1(\alpha)\mathcal{G}_1(\alpha')] = \sum_{k=2}^{\infty} a_k(\alpha)a_k(\alpha')$; this is also the covariance structure of $\bar{\mathcal{G}}_1$, as

$$E[\bar{\mathcal{G}}_1(\alpha)\bar{\mathcal{G}}_1(\alpha')] = \sum_{k=2}^{\infty} a_k(\alpha)a_k(\alpha')E[Z_k^2] = \sum_{k=2}^{\infty} a_k(\alpha)a_k(\alpha').$$

This completes the proof.

(ii) Likewise, we verify that for each α, $\alpha' \in A$, $E[\mathcal{G}_2(\alpha)\mathcal{G}_2(\alpha')] = E[\bar{\mathcal{G}}_2(\alpha)\bar{\mathcal{G}}_2(\alpha')]$. For this, note that

$$E[\bar{\mathcal{G}}_2(\alpha)\bar{\mathcal{G}}_2(\alpha')] = \sum_{k=1}^{\infty} b_k(\alpha)b_k(\alpha') = \frac{N_1(\alpha, \alpha')}{D_1(\alpha)^{1/2}D_1(\alpha')^{1/2}},$$

where

$$D_1(\alpha) = q^2\left[\frac{q^2}{(1 - q^2)} - \left(\frac{6}{\pi^2}\right)\ln(\alpha)^2\right] \quad \text{and}$$

$$N_1(\alpha, \alpha') \equiv qq'\sum_{k=1}^{\infty}\left\{q^k - \frac{6}{(\pi^2 k)}\ln(\alpha)\right\}\left\{q'^k - \frac{6}{(\pi^2 k)}\ln(\alpha')\right\}.$$

Therefore,

$$N_1(\alpha, \alpha') = qq'\left(\sum_{k=1}^{\infty}(qq')^k - \frac{6}{\pi^2}\ln(\alpha')\sum_{k=1}^{\infty}\frac{q^k}{k} - \frac{6}{\pi^2}\ln(\alpha)\sum_{k=1}^{\infty}\frac{q'^k}{k} + \frac{36}{\pi^4}\ln(\alpha)\ln(\alpha')\sum_{k=1}^{\infty}\frac{1}{k^2}\right).$$

Now $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$, and $\sum_{k=1}^{\infty} q^k/k = \sum_{k=1}^{\infty} \frac{1}{k}q^k = \ln(\alpha)$ for $\alpha \geq 1/2$ (see formulas 19.19 and 20.20 in Spiegel, 1968), so that

$$N_1(\alpha, \alpha') = qq'\left\{\frac{qq'}{(1 - qq')} - \left(\frac{6}{\pi^2}\right)\ln(\alpha)\ln(\alpha')\right\},$$

implying the desired equality, $E[\mathcal{G}_2(\alpha)\mathcal{G}_2(\alpha')] = E[\bar{\mathcal{G}}_2(\alpha)\,\bar{\mathcal{G}}_2(\alpha')]$. This completes the proof.

(iii) Before proving the main claims, we first show that $6/\pi^2 - \xi^*/\gamma^{*2} \geq 0$, so that its square root exists. By the definition of ξ^*, it follows that

$$\xi^*/\gamma^{*2} = \frac{1}{\pi^2/6 + \gamma^{*2}c^*},$$

where $c^* \equiv \text{var}[\phi_t^*] - \text{cov}[\phi_t^*, d_t^*]'(\text{var}[d_t^*])^{-1}\text{cov}[\phi_t^*, d_t^*] \geq 0$. It follows immediately that $6/\pi^2 - \xi^*/\gamma^{*2} \geq 0$.

(a) We show that for each $\alpha, \alpha' \in A$, $E[\mathcal{G}_3^*(\alpha)\mathcal{G}_3^*(\alpha')] = E[\bar{\mathcal{G}}_3^*(\alpha)\bar{\mathcal{G}}_3^*(\alpha')]$ as before. Note that

$$E[\bar{\mathcal{G}}_3^*(\alpha)\bar{\mathcal{G}}_3^*(\alpha')] = \sum_{k=0}^{\infty} c_k(\alpha, \theta^*)c_k(\alpha', \theta^*) = \frac{N_2(\alpha, \alpha', \theta^*)}{D_2(\alpha)^{1/2}D_2(\alpha')^{1/2}},$$

where

$$N_2(\alpha, \alpha', \theta^*) \equiv qq'\{N_1(\alpha, \alpha') + \left(6/\pi^2 - \xi^*/\gamma^{*2}\right)\ln(\alpha)\ln(\alpha')\}, \quad \text{and}$$

$$D_2(\alpha) \equiv q^2\left[\frac{q^2}{(1-q^2)} - \left(\frac{\xi^*}{\gamma^{*2}}\right)\ln(\alpha)^2\right].$$

Thus, plugging the definition of $N_1(\alpha, \alpha')$ into $N_2(\alpha, \alpha', \theta^*)$ yields

$$N_2(\alpha, \alpha', \theta^*) = \left\{\frac{qq'}{(1-qq')} - \left(\frac{\xi^*}{\gamma^{*2}}\right)\ln(\alpha)\ln(\alpha')\right\},$$

so that the desired equality $E[\mathcal{G}_3^*(\alpha)\mathcal{G}_3^*(\alpha')] = E[\bar{\mathcal{G}}_3^*(\alpha)\ \bar{\mathcal{G}}_3^*(\alpha')]$ follows.

(b) For this, we show that $\sup_{\alpha\in A}|\widehat{\mathcal{G}}_{3,n}(\alpha) - \widetilde{\mathcal{G}}_{3,n}(\alpha)| = o_p(1)$, and $\sup_{\alpha\in A}|\widetilde{\mathcal{G}}_{3,n}(\alpha) - \bar{\mathcal{G}}_3^*(\alpha)| = o_p(1)$, where for each $\alpha \in A$,

$$\widetilde{\mathcal{G}}_{3,n}(\alpha) \equiv \sum_{k=0}^{\infty} c_k(\alpha, \theta^*, \hat{\theta}_n)Z_k;$$

$$c_0(\alpha, \theta^*, \hat{\theta}_n) \equiv \left\{\frac{6}{\pi^2} - \frac{\xi^*}{\gamma^{*2}}\right\}^{1/2}\frac{q\ln(\alpha)}{Q(\alpha; \hat{\xi}_n, \hat{\gamma}_n)^{1/2}};$$

for $k = 1, 2, \cdots$,

$$c_k(\alpha, \theta^*, \hat{\theta}_n) \equiv q\frac{\left\{q^k - \frac{6\ln(\alpha)}{(\pi^2 k)}\right\}}{Q(\alpha; \hat{\xi}_n, \hat{\gamma}_n)^{1/2}}; \quad \text{and}$$

$$Q(\alpha;\hat{\xi}_n,\hat{\gamma}_n) \equiv q^2\left\{\frac{(\alpha-1)^2}{(2\alpha-1)} - \left(\frac{\hat{\xi}_n}{\hat{\gamma}_n^2}\right)\ln(\alpha)^2\right\}.$$

The desired result then follows, as

$$\sup_{\alpha\in A}|\widehat{\mathcal{G}}_{3,n}(\alpha) - \bar{\mathcal{G}}_3^*(\alpha)| \le \sup_{\alpha\in A}|\widehat{\mathcal{G}}_{3,n}(\alpha) - \widetilde{\mathcal{G}}_{3,n}(\alpha)| + \sup_{\alpha\in A}|\widetilde{\mathcal{G}}_{3,n}(\alpha) - \bar{\mathcal{G}}_3^*(\alpha)| = o_p(1).$$

First, we note that

$$\widehat{\mathcal{G}}_{3,n}(\alpha) - \widetilde{\mathcal{G}}_{3,n}(\alpha) = q\ln(\alpha)\left\{\left[\frac{6}{\pi^2} - \frac{\hat{\xi}_n}{\hat{\gamma}_n^2}\right]^{1/2} - \left[\frac{6}{\pi^2} - \frac{\xi^*}{\gamma^{*2}}\right]^{1/2}\right\}\frac{Z_0}{Q(\alpha,\hat{\xi}_n,\hat{\gamma}_n)^{1/2}}.$$

Also,

$$\sup_{\alpha\in A}\left|\ln(\alpha)\left(\frac{\alpha-1}{\alpha}\right)\left[\frac{(\alpha-1)^2}{\alpha^2}\left[\frac{(\alpha-1)^2}{(2\alpha-1)} - \left(\frac{\hat{\xi}_n}{\hat{\gamma}_n^2}\right)\ln(\alpha)^2\right]\right]^{-1/2}\right| = O_p(1)$$

as a consequence of the functional form with respect to α, the fact that $(\hat{\xi}_n,\hat{\gamma}_n) = (\xi^*,\gamma^*) + o_p(1)$, and the fact that A is a compact set. Next, we separately consider the two cases (i) $6/\pi^2 - \xi^*/\gamma^{*2} = 0$ and (ii) $6/\pi^2 - \xi^*/\gamma^{*2} > 0$. (i) If $6/\pi^2 - \xi^*/\gamma^{*2} = 0$, then

$$\left|\left[\left\{\frac{6}{\pi^2} - \frac{\hat{\xi}_n}{\hat{\gamma}_n^2}\right\}^{1/2} - \left\{\frac{6}{\pi^2} - \frac{\xi^*}{\gamma^{*2}}\right\}^{1/2}\right]Z_0\right| = \left|\left\{\frac{6}{\pi^2} - \frac{\hat{\xi}_n}{\hat{\gamma}_n^2}\right\}^{1/2}Z_0\right| = o_p(1),$$

because $(\hat{\xi}_n,\hat{\gamma}_n) = (\xi^*,\gamma^*) + o_p(1)$ and $Z_0 \sim N(0,1)$. (ii) If $6/\pi^2 - \xi^*/\gamma^{*2} > 0$, then $\{6/\pi^2 - \hat{\xi}_n/\hat{\gamma}_n^2\}^{1/2}$ is continuous as a function of $(\hat{\xi}_n,\hat{\gamma}_n)$; it follows that

$$\left|\left[\left\{\frac{6}{\pi^2} - \frac{\hat{\xi}_n}{\hat{\gamma}_n^2}\right\}^{1/2} - \left\{\frac{6}{\pi^2} - \frac{\xi^*}{\gamma^{*2}}\right\}^{1/2}\right]Z_0\right| = o_p(1),$$

for the same reasons as before. We conclude that $\sup_{\alpha\in A}|\widehat{\mathcal{G}}_{3,n}(\alpha) - \widetilde{\mathcal{G}}_{3,n}(\alpha)| = o_p(1)$.

Next, we have that

$$\widetilde{\mathcal{G}}_{3,n}(\alpha) - \bar{\mathcal{G}}_3^*(\alpha) = \left\{Q(\alpha;\hat{\xi}_n,\hat{\gamma}_n)^{-1/2} - Q(\alpha;\xi^*,\gamma^*)^{-1/2}\right\} \times \left\{\left[\frac{6}{\pi^2} - \frac{\xi^*}{\gamma^{*2}}\right]^{1/2} q\ln(\alpha)Z_0 + \sum_{k=1}^{\infty} q\left[q^k - \frac{6}{(\pi^2 k)}\ln(\alpha)\right]Z_k\right\},$$

and we note that

$$\sup_{\alpha\in A}\left|\left(\frac{6}{\pi^2} - \frac{\xi^*}{\gamma^{*2}}\right)^{1/2}\left[\frac{\alpha-1}{\alpha}\right]\ln(\alpha)Z_0 + \sum_{k=1}^{\infty}\left\{\left[\frac{\alpha-1}{\alpha}\right]^k - \frac{6}{\pi^2 k}\ln(\alpha)\right\}\left[\frac{\alpha-1}{\alpha}\right]Z_k\right| = O_p(1),$$

because the given function is a continuous Gaussian process whose variance is uniformly bounded by

$$\sup_{\alpha\in A}\left|\frac{(\alpha-1)^2}{\alpha^2}\left\{\frac{(\alpha-1)^2}{(2\alpha-1)} - \left(\frac{\xi^*}{\gamma^{*2}}\right)\ln(\alpha)^2\right\}\right|.$$

This is finite given that $\inf A > 1/2$ and that A is a compact set. Also,

$$\sup_{\alpha\in A}\left|\frac{\alpha-1}{\alpha}\right| \times \left|\left\{\frac{(\alpha-1)^2}{2\alpha-1} - \frac{\hat{\xi}_n}{\hat{\gamma}_n^2}\ln(\alpha)^2\right\}^{-1/2} - \left\{\frac{(\alpha-1)^2}{2\alpha-1} - \frac{\xi^*}{\gamma^{*2}}\ln(\alpha)^2\right\}^{-1/2}\right| = o_p(1),$$

because $\inf A > 1/2$; A is compact; as a function of α, the given function is uniformly continuous; and $(\hat{\xi}_n, \hat{\gamma}_n) = (\xi^*, \gamma^*) + o_p(1)$. We thus have $\sup_{\alpha\in A}|\widetilde{\mathcal{G}}_{3,n}(\alpha) - \bar{\mathcal{G}}_3^*(\alpha)| = o_p(1)$, and the proof is complete. ■

We prove the consistency of the LR test statistic using the following supplementary lemma, which rephrases lemma A1 in Cho and White (2007) for the present context.

Lemma A1: *Given Assumptions A1, A2, and A3*, $\sup_{(\pi,\alpha_1,\alpha_2,\beta,\gamma)}|n^{-1}\sum \ell_t(\pi,\alpha_1,\alpha_2,\beta,\gamma) - E[\ell_t(\pi,\alpha_1,\alpha_2,\beta,\gamma)]| \stackrel{\text{a.s.}}{\to} 0$.

Proof of Lemma A1: First, note that ℓ_t is differentiable a.s. because of A2(i) and the definitions of $f(y|X_t;\cdot,\cdot,\cdot)$ and $m(y|X_t;\cdot,\cdot,\cdot,\cdot)$. Second, for some positive, stationary, and ergodic random variable, M_t, $\|\nabla_{(\pi,\alpha_1,\alpha_2,\beta,\gamma)}\ell_t(\pi,\alpha_1,\alpha_2,\beta,\gamma)\|_\infty < M_t$ by A3. Third, therefore, for each $(\pi,\alpha_1,\alpha_2,\beta,\gamma)$ and $(\tilde{\pi},\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\beta},\tilde{\gamma})$, it follows that

$$|\ell_t(\pi,\alpha_1,\alpha_2,\beta,\gamma) - \ell_t(\tilde{\pi},\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\beta},\tilde{\gamma})| \le M_t\|(\pi,\alpha_1,\alpha_2,\beta',\gamma)' - (\tilde{\pi},\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\beta}',\tilde{\gamma})'\|.$$

This also implies that

$$|n^{-1}\sum \ell_t(\pi,\alpha_1,\alpha_2,\beta,\gamma) - n^{-1}\sum \ell_t(\tilde{\pi},\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\beta}',\tilde{\gamma})| \\ \leq n^{-1}\sum M_t \|(\pi,\alpha_1,\alpha_2,\beta,\gamma)' - (\tilde{\pi},\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\beta}',\tilde{\gamma})'\|.$$

Fourth, we now apply the ergodic theorem, so that for any $\omega \in F$, $P(F) = 1$, and $\varepsilon > 0$, there is an $n^*(\omega,\varepsilon)$ such that if $n \geq n^*(\omega,\varepsilon)$, then $|n^{-1}\sum M_t - E[M_t]| \leq \varepsilon$ so that $n^{-1}\sum M_t \leq E[M_t] + \varepsilon$. Fifth, for the same ε, if $\delta \equiv \varepsilon/(E[M_t]+\varepsilon)$ and $n \geq n^*(\omega,\varepsilon)$, then

$$n^{-1}\sum M_t \|(\pi,\alpha_1,\alpha_2,\beta',\gamma)' - (\tilde{\pi},\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\beta}',\tilde{\gamma})'\| \leq n^{-1}\sum M_t\delta \\ = n^{-1}\sum M_t\varepsilon/(\varepsilon + E[M_t]) \leq \varepsilon,$$

whenever $\|(\pi,\alpha_1,\alpha_2,\beta',\gamma)' - (\tilde{\pi},\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\beta}',\tilde{\gamma})'\| \leq \delta$. That is, for any $\omega \in F$, $P(F) = 1$ and $\varepsilon > 0$, there are $n^*(\omega,\varepsilon)$ and δ such that if $n \geq n^*(\omega,\varepsilon)$ and $\|(\pi,\alpha_1,\alpha_2,\beta,\gamma)' - (\tilde{\pi},\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\beta}',\tilde{\gamma})'\| \leq \delta$, then $|n^{-1}\sum \ell_t(\pi,\alpha_1,\alpha_2,\beta,\gamma) - n^{-1}\sum \ell_t(\tilde{\pi},\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\beta}',\tilde{\gamma})| < \varepsilon$, which means that $\{n^{-1}\sum \ell_t\}_{n^*(\omega,\varepsilon)}^{\infty}$ is equicontinuous. Thus, $n^{-1}\sum \ell_t$ converges to $E[\ell_t]$ uniformly on $[0,1]\times A\times A\times B\times\Gamma$ almost surely by Rudin (1976, p.168). ■

Proof of Theorem 3: By the properties of the Kullback-Leibler information criterion, $E[\ln f_a(Y_t|X_t;\pi^*,\alpha_1^*,\alpha_2^*,\beta^*,\gamma^*)] > E[\ln f(Y_t|X_t;\zeta^*,\beta_{o,a}^*,\gamma_{o,a}^*)]$ under $\mathcal{H}_a$. Therefore, applying Lemma A1 implies that for any $\omega \in F$, $P(F) = 1$ and $\varepsilon > 0$, there exists $n^*(\omega,\varepsilon)$ such that if $n \geq n^*(\omega,\varepsilon)$ then $|A_{1n}| < \varepsilon$, $|A_{2n}| < \varepsilon$, $|B_{1n}| < \varepsilon$, and $|B_{2n}| < \varepsilon$, implying the consistency of the MLE, where

$$A_{1n} \equiv n^{-1}\sum \ln f_a(Y_t|X_t;\hat{\pi}_n,\hat{\alpha}_{1n},\hat{\alpha}_{2n},\hat{\beta}_{an},\hat{\gamma}_{an}) - n^{-1}\sum \ln f_a(Y_t|X_t;\pi^*,\alpha_1^*,\alpha_2^*,\beta^*,\gamma^*);$$

$$A_{2n} \equiv n^{-1}\sum \ln f_a(Y_t|X_t;\pi^*,\alpha_1^*,\alpha_2^*,\beta^*,\gamma^*) - E[\ln f_a(Y_t|X_t;\pi^*,\alpha_1^*,\alpha_2^*,\beta^*,\gamma^*)];$$

$$B_{1n} \equiv n^{-1}\sum \ln f(Y_t|X_t;\hat{\delta}_n,\hat{\beta}_n,\hat{\gamma}_n) - n^{-1}\sum \ln f(Y_t|X_t;\zeta^*,\beta_{o,a}^*,\gamma_{o,a}^*); \quad \text{and}$$

$$B_{2n} \equiv n^{-1}\sum \ln f(Y_t|X_t;\zeta^*,\beta_{o,a}^*,\gamma_{o,a}^*) - E[\ln f(Y_t|X_t;\zeta^*,\beta_{o,a}^*,\gamma_{o,a}^*)].$$

This implies that $|(A_{1n}+A_{2n}) - (B_{1n}+B_{2n})| \leq 4\varepsilon$, so that if we let

$$\hat{\Upsilon}_n \equiv n^{-1}\sum \ln f_a(Y_t|X_t;\hat{\pi}_n,\hat{\alpha}_{1n},\hat{\alpha}_{2n},\hat{\beta}_{an},\hat{\gamma}_{an}) - n^{-1}\sum \ln f(Y_t|X_t;\hat{\delta}_n,\hat{\beta}_n,\hat{\gamma}_n) \quad \text{and}$$

$$\Upsilon^* \equiv E[\ln f_a(Y_t|X_t;\pi^*,\alpha_1^*,\alpha_2^*,\beta^*,\gamma^*)] - E[\ln f(Y_t|X_t;\zeta^*,\beta_{o,a}^*,\gamma_{o,a}^*)],$$

then it follows that $\Upsilon^* - 4\varepsilon \leq \hat{\Upsilon}_n \leq \Upsilon^* + 4\varepsilon$. Thus, for some $\delta_1 \in (0, \Upsilon^* - 4\varepsilon)$ and $\delta_2 \in (\Upsilon^* + 4\varepsilon, \infty)$, if $n > n^*(\omega, \varepsilon)$, then $\delta_1 < \hat{\Upsilon}_n < \delta_2$. That is, for any $\omega \in F$, $P(F) = 1$ and $\varepsilon > 0$, there are $n^*(\omega, \varepsilon)$, δ_1, and δ_2 such that if $n \geq n^*(\omega, \varepsilon)$, then $0 < 2\delta_1 < n^{-1}LR_n < 2\delta_2 < \infty$, using the fact that $LR_n = 2n\hat{\Upsilon}_n$. Thus, $LR_n = O_p(n)$ but not $o_p(n)$. This is the desired result. ■

References

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